Outline

• Petri nets
  – Introduction
  – Examples
  – Properties
  – Analysis techniques
Petri Nets (PNs)

• Model introduced by C.A. Petri in 1962
  – Ph.D. Thesis: “Communication with Automata”

• Applications: distributed computing, manufacturing, control, communication networks, transportation…

• PNs describe explicitly and graphically:
  – sequencing/causality
  – conflict/non-deterministic choice
  – concurrency

• Basic PN model
  – Asynchronous model (partial ordering)
  – Main drawback: no hierarchy
Example: Synchronization at single track rail segment

- "Preconditions"
Playing the "token game"
Conflict for resource “track”
Petri Net Graph

- Bipartite weighted directed graph:
  - Places: circles
  - Transitions: bars or boxes
  - Arcs: arrows labeled with weights
- Tokens: black dots
Petri Net

- A PN \((N,M_0)\) is a Petri Net Graph \(N\)
  - places: represent distributed state by holding tokens
    - marking (state) \(M\) is an \(n\)-vector \((m_1,m_2,m_3\ldots)\), where \(m_i\) is the non-negative number of tokens in place \(p_i\).
    - initial marking \((M_0)\) is initial state
  - transitions: represent actions/events
    - enabled transition: enough tokens in predecessors
    - firing transition: modifies marking
- ...and an initial marking \(M_0\).

Places/Transitions: conditions/events
Transition firing rule

- A marking is changed according to the following rules:
  - A transition is enabled if there are enough tokens in each input place
  - An enabled transition may or may not fire
  - The firing of a transition modifies marking by consuming tokens from the input places and producing tokens in the output places
Concurrency, causality, choice

\[ t_1 \]

\[ t_2 \]

\[ t_3 \]

\[ t_4 \]

\[ t_5 \]

\[ t_6 \]
Concurrency, causality, choice

Concurrent processes:
- `t1` to `t2`
- `t3` to `t4`

Decision points:
- `t5`
- `t6`
Concurrency, causality, choice

Causality, sequencing

\[ t1 \rightarrow \text{event} \rightarrow t2 \]

\[ t3 \rightarrow \text{event} \rightarrow t4 \rightarrow t5 \rightarrow t6 \]
Concurrency, causality, choice

-diagram-
Concurrency, causality, choice

Choice, conflict
Communication Protocol

P1

Send msg

Receive Ack

Send Ack

Receive msg

P2
Communication Protocol

P1

Send msg

Receive Ack

Send Ack

Receive msg

P2

Receive Ack

Send Ack
Communication Protocol

P1

Send msg

Receive Ack

Send Ack

Receive msg

P2

Receive Ack

Send Ack
Communication Protocol

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Receive Ack

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P2

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Communication Protocol

P1

Send msg

Receive Ack

Send Ack

Receive msg

P2

Receive Ack

Send Ack
Communication Protocol

P1
Send msg
Receive Ack
Send Ack
Receive msg
P2

Receive Ack
Send Ack
Producer-Consumer Problem
Producer-Consumer Problem
Producer-Consumer Problem

Produce

Buffer

Consume
Producer-Consumer Problem

![Diagram of the Producer-Consumer Problem with a buffer between production and consumption.](image-url)
Producer-Consumer Problem

![Diagram of Producer-Consumer Problem]
Producer-Consumer Problem

Produce

Buffer

Consume
Producer-Consumer Problem
Producer-Consumer Problem
Producer-Consumer Problem

Produce

Buffer

Consume
Producer-Consumer Problem
Producer-Consumer Problem
Producer-Consumer Problem
Producer-Consumer Problem
Producer-Consumer Problem
Producer-Consumer with priority

Consumer B can consume only if buffer A is empty

Inhibitor arcs
PN properties

- **Behavioral**: depend on the initial marking (most interesting)
  - Reachability
  - Boundedness
  - Schedulability
  - Liveness
  - Conservation
- **Structural**: do not depend on the initial marking (often too restrictive)
  - Consistency
  - Structural boundedness
Reachability

- Marking $M$ is reachable from marking $M_0$ if there exists a sequence of firings $\sigma = M_0 \, t_1 \, M_1 \, t_2 \, M_2 \ldots \, M$ that transforms $M_0$ to $M$.
- The reachability problem is decidable.

\[
\begin{align*}
M_0 & = (1,0,1,0) \\
M & = (1,1,0,0)
\end{align*}
\]

\[
\begin{align*}
M_0 & = (1,0,1,0) & \quad \text{t3} \\
M_1 & = (1,0,0,1) & \quad \text{t2} \\
M & = (1,1,0,0)
\end{align*}
\]
Liveness

- Liveness: from any marking any transition can become fireable
  - Liveness implies deadlock freedom, not vice versa

Not live
Liveness

- Liveness: from any marking any transition can become fireable
  - Liveness implies deadlock freedom, not vice versa

![Diagram of a Petri net with a marking and transitions, indicating liveness and non-liveness]
Liveness

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  - Liveness implies deadlock freedom, not vice versa
Liveness

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  - Liveness implies deadlock freedom, not vice versa
Boundedness

- **Boundedness**: the number of tokens in any place cannot grow indefinitely
  - (1-bounded also called *safe*)
  - Application: places represent buffers and registers (check there is no overflow)

---

![Diagram of a Petri net with places and transitions illustrating boundedness and unboundedness.](image-url)
Boundedness

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Conservation

- Conservation: the total number of tokens in the net is constant

Not conservative
Conservation

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Not conservative
Conservation

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Conservational
Analysis techniques

• Structural analysis techniques
  – Incidence matrix
  – T- and S- Invariants

• State Space Analysis techniques
  – Coverability Tree
  – Reachability Graph
Incidence Matrix

- Necessary condition for marking $M$ to be reachable from initial marking $M_0$:
  
  there exists firing vector $v$ s.t.:

  $$M = M_0 + A \cdot v$$
State equations

- E.g. reachability of $M = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ from $M_0 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$

\[
A = \begin{bmatrix}
-1 & 0 & 0 \\
1 & 1 & -1 \\
0 & -1 & 1
\end{bmatrix}
\]

\[
v_1 = \begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix}
\]

\[
v_1 = \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} + \begin{bmatrix}
-1 \\
1 \\
0
\end{bmatrix}
\]

but also $v_2 = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}^T$ or any $v_k = \begin{bmatrix} 1 & (k) & (k+1) \end{bmatrix}^T$
Necessary Condition only

Firing vector: (1,2,2)  
Deadlock!!
State equations and invariants

- Solutions of $Ax = 0$ (in $M = M_0 + Ax$, $M = M_0$)
  
  T-invariants
  - sequences of transitions that (if fireable) bring back to original marking
  - periodic schedule in SDF
  - e.g. $x = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T$

\[
A = \begin{bmatrix}
-1 & 0 & 0 \\
1 & 1 & -1 \\
0 & -1 & 1
\end{bmatrix}
\]
Application of T-invariants

- **Scheduling**
  - Cyclic schedules: need to return to the initial state

T-invariant: \((1,1,1,1,1)\)
Schedule: \(i \ast k2 \ast k1 + o\)
State equations and invariants

- Solutions of $yA = 0$

  S-invariants
  - sets of places whose weighted total token count does not change after the firing of any transition ($yM = yM'$)
  - e.g. $y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$

\[
A^T = \begin{bmatrix}
-1 & 1 & 0 \\
0 & 1 & -1 \\
0 & -1 & 1
\end{bmatrix}
\]
Application of S-invariants

• **Structural Boundedness:** bounded for any finite initial marking $M_0$

• **Existence of a positive S-invariant is sufficient condition for structural boundedness**
  - initial marking is finite
  - weighted token count does not change
Summary of algebraic methods

- Extremely efficient
  (polynomial in the size of the net)
- Generally provide only necessary or sufficient information
- Excellent for ruling out some deadlocks or otherwise dangerous conditions
- Can be used to infer structural boundedness
Coverability Tree

• Build a (finite) tree representation of the markings

Karp-Miller algorithm

• Label initial marking \( M_0 \) as the root of the tree and tag it as \textit{new}

• While new markings exist do:
  – select a new marking \( M \)
  – if \( M \) is identical to a marking on the path from the root to \( M \), then tag \( M \) as \textit{old} and go to another new marking
  – if no transitions are enabled at \( M \), tag \( M \) \textit{dead-end}
  – while there exist enabled transitions at \( M \) do:
    – obtain the marking \( M' \) that results from firing \( t \) at \( M \)
    – on the path from the root to \( M \) if there exists a marking \( M'' \) such that \( M'(p) \geq M''(p) \) for each place \( p \) and \( M' \) is different from \( M'' \), then replace \( M'(p) \) by \( \omega \) for each \( p \) such that \( M'(p) > M''(p) \)
    – introduce \( M' \) as a node, draw an arc with label \( t \) from \( M \) to \( M' \) and tag \( M' \) as \textit{new}. 
Coverability Tree

- Boundedness is decidable with *coverability tree*
Coverability Tree

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Coverability Tree

• Boundedness is decidable
  with *coverability tree*

```
<table>
<thead>
<tr>
<th>p1</th>
<th>t1</th>
<th>p2</th>
<th>t2</th>
<th>p3</th>
<th>t3</th>
<th>p4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
```

```
1000
0100
0011
```
Coverability Tree

- Boundedness is decidable with *coverability tree*
Coverability Tree

- Boundedness is decidable with *coverability tree*

![Coverability Tree Diagram]

- States: p1, p2, p3, p4
- Transitions: t1, t2, t3

- Paths:
  - 1000 \(\xrightarrow{t1} 0100 \xrightarrow{t3} 0011 \xrightarrow{t2} 0100\)
Coverability Tree

- Is (1) reachable from (0)?

```
  t1   p1   t2
   ┌───┐   ┌───┐
   │   │   │   │
   v   v   v   v
  t1_2 p1_2 t2
```

64
Coverability Tree

- Is (1) reachable from (0)?
Coverability Tree

- Is (1) reachable from (0)?

\[ \begin{align*}
\text{t1} & \quad \text{p1} & \quad \text{t2} \\
0 & \downarrow \text{t1} & \omega \quad (0) \rightarrow (1) \rightarrow (2) \ldots \\
\text{t1} & \quad \text{p1} & \quad \text{t2} \\
0 & \downarrow \text{t1} & \omega \quad (0) \rightarrow (2) \rightarrow (0) \ldots
\end{align*} \]
Coverability Tree

• Is (1) reachable from (0)?

• Cannot solve the reachability problem
Reachability graph

- For bounded nets the Coverability Tree is called Reachability Tree since it contains all possible reachable markings
Reachability graph

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Reachability graph

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Subclasses of Petri nets

- Reachability analysis is too expensive
- State equations give only partial information
- Some properties are preserved by reduction rules
  e.g. for liveness and safeness

- Even reduction rules only work in some cases
- Must restrict class in order to prove stronger results
Marked Graphs

- Every place has at most 1 predecessor and 1 successor transition
- Models only causality and concurrency (no conflict)
State Machines

- Every transition has at most 1 predecessor and 1 successor place
- Models only causality and conflict
  - (no concurrency, no synchronization of parallel activities)
Free-Choice Petri Nets (FCPN)

Free-Choice (FC)

- every transition after choice has exactly 1 predecessor place

Confusion (not-Free-Choice)  Extended Free-Choice

Free-Choice: the outcome of a choice depends on the value of a token (abstracted non-deterministically) rather than on its arrival time.
Free-Choice nets

- Introduced by Hack (‘72)
- Extensively studied by Best (‘86) and Desel and Esparza (‘95)
- Can express concurrency, causality and choice without confusion
- Very strong structural theory
  - necessary and sufficient conditions for liveness and safeness, based on decomposition
  - exploits duality between MG and SM
MG (& SM) decomposition

• An Allocation is a control function that chooses which transition fires among several conflicting ones (A: P → T).

• Eliminate the subnet that would be inactive if we were to use the allocation...

• Reduction Algorithm
  – Delete all unallocated transitions
  – Delete all places that have all input transitions already deleted
  – Delete all transitions that have at least one input place already deleted

• Obtain a Reduction (one for each allocation) that is a conflict free subnet
MG reduction and cover

- Choose one successor for each conflicting place:
MG reduction and cover

- Choose one successor for each conflicting place:
MG reduction and cover

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MG reduction and cover

- Choose one successor for each conflicting place:
MG reductions

- The set of all reductions yields a cover of MG components (T-invariants)
MG reductions

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Hack’s theorem (‘72)

• Let N be a Free-Choice PN:
  – N has a live and safe initial marking (well-formed)
  if and only if
  – every MG reduction is strongly connected and not empty, and
    the set of all reductions covers the net
  – every SM reduction is strongly connected and not empty, and
    the set of all reductions covers the net
Hack’s theorem

- Example of non-live (but safe) FCN
Hack’s theorem

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Summary of LSFC nets

• Largest class for which structural theory really helps
• Structural component analysis may be expensive
  (exponential number of MG and SM components in the worst case)
• But…
  – number of MG components is generally small
  – FC restriction simplifies characterization of behavior
Petri Net extensions

• Add interpretation to tokens and transitions
  – Colored nets (tokens have value)

• Add time
  – Time/timed Petri Nets (deterministic delay)
    – type (duration, delay)
    – where (place, transition)
  – Stochastic PNs (probabilistic delay)
  – Generalized Stochastic PNs (timed and immediate transitions)

• Add hierarchy
  – Place Charts Nets
PNs Summary

• PN Graph: places (buffers), transitions (actions), tokens (data)
• Firing rule: transition enabled if there are enough tokens in each input place
• Properties
  – Structural (consistency, structural boundedness…)
  – Behavioral (reachability, boundedness, liveness…)
• Analysis techniques
  – Structural (only CN or CS): State equations, Invariants
  – Behavioral: coverability tree
• Reachability
• Subclasses: Marked Graphs, State Machines, Free-Choice PNs
References

• T. Murata Petri Nets: Properties, Analysis and Applications
• http://www.informatik.uni-hamburg.de/TGI/PetriNets/!!