Mathematical Model of Passive Bipedal Walker (3D)

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Energy Equations

```
l = a + b; vectH = l Sin[ϕs[t]] {-Sin[θs[t]], Cos[θs[t]], 0} θs'[t] +
  l {Cos[θs[t]] Cos[ϕs[t]], Sin[θs[t]] Cos[ϕs[t]], -Sin[ϕs[t]]} ϕs'[t];
vectS = a Sin[ϕs[t]] {-Sin[θs[t]], Cos[θs[t]], 0} θs'[t] +
   a {Cos[θs[t]] Cos[ϕs[t]], Sin[θs[t]] Cos[ϕs[t]], -Sin[ϕs[t]]} ϕs'[t];
absVectH = FullSimplify[Norm[vectH], {θs[t] ∈ Reals, ϕs[t] ∈ Reals,
   θs'[t] ∈ Reals, ϕs'[t] ∈ Reals, a ∈ Reals, b ∈ Reals}]
absVectS = FullSimplify[Norm[vectS], {θs[t] ∈ Reals, ϕs[t] ∈ Reals, θs'[t] ∈ Reals, ϕs'[t] ∈ Reals, a ∈ Reals}]
```

$\text{Abs}[a+b]\sqrt{\text{Sin}[\phi s[t]]^2\,\theta s'[t]^2+\phi s'[t]^2}$

$\text{Abs}[a]\sqrt{\text{Sin}[\phi s[t]]^2\,\theta s'[t]^2+\phi s'[t]^2}$

```
vectNS = D[{l Cos[θs[t]] Sin[ϕs[t]] - b Cos[θns[t]] Sin[ϕns[t]],
    l Sin[θs[t]] Sin[ϕs[t]] - b Sin[θns[t]] Sin[ϕns[t]], l Cos[ϕs[t]] - b Cos[ϕns[t]]}, t];
absVectNS = FullSimplify[Norm[vectNS], {vectNS ∈ Reals}]
```

```
√((b Sin[θns[t]] Sin[ϕns[t]] θns′[t] - (a + b) Sin[θs[t]] Sin[ϕs[t]] θs′[t] -
      b Cos[θns[t]] Cos[ϕns[t]] ϕns′[t] + (a + b) Cos[θs[t]] Cos[ϕs[t]] ϕs′[t])^2 +
   (b Cos[θns[t]] Sin[ϕns[t]] θns′[t] - (a + b) Cos[θs[t]] Sin[ϕs[t]] θs′[t] + b Cos[ϕns[t]] Sin[θns[t]] ϕns′[t] -
      (a + b) Cos[ϕs[t]] Sin[θs[t]] ϕs′[t])^2 + (b Sin[ϕns[t]] ϕns′[t] - (a + b) Sin[ϕs[t]] ϕs′[t])^2)
```

```
U = m g a Cos[ϕs[t]] + (M + m) g l Cos[ϕs[t]] - m g b Cos[ϕns[t]];
K = FullSimplify[1 / 2 M Power[absVectH, 2] + 1 / 2 m Power[absVectS, 2] + 1 / 2 m Power[absVectNS, 2],
  {θs[t] ∈ Reals, ϕs[t] ∈ Reals, θns[t] ∈ Reals, ϕns[t] ∈ Reals, θs'[t] ∈ Reals,
   ϕs'[t] ∈ Reals, θns'[t] ∈ Reals, ϕns'[t] ∈ Reals, a ∈ Reals, b ∈ Reals, m ∈ Reals, M ∈ Reals}]
```

```
1/2 (a^2 m (Sin[ϕs[t]]^2 θs′[t]^2 + ϕs′[t]^2) + (a + b)^2 M (Sin[ϕs[t]]^2 θs′[t]^2 + ϕs′[t]^2) +
   m ((b Sin[θns[t]] Sin[ϕns[t]] θns′[t] - (a + b) Sin[θs[t]] Sin[ϕs[t]] θs′[t] -
         b Cos[θns[t]] Cos[ϕns[t]] ϕns′[t] + (a + b) Cos[θs[t]] Cos[ϕs[t]] ϕs′[t])^2 +
      (b Cos[θns[t]] Sin[ϕns[t]] θns′[t] - (a + b) Cos[θs[t]] Sin[ϕs[t]] θs′[t] + b Cos[ϕns[t]] Sin[θns[t]] ϕns′[t] -
         (a + b) Cos[ϕs[t]] Sin[θs[t]] ϕs′[t])^2 + (b Sin[ϕns[t]] ϕns′[t] - (a + b) Sin[ϕs[t]] ϕs′[t])^2))
```

Lagrangian Formulation

```
L = K - U
```

```
b g m Cos[ϕns[t]] - a g m Cos[ϕs[t]] - (a + b) g (m + M) Cos[ϕs[t]] +
 1/2 (a^2 m (Sin[ϕs[t]]^2 θs′[t]^2 + ϕs′[t]^2) + (a + b)^2 M (Sin[ϕs[t]]^2 θs′[t]^2 + ϕs′[t]^2) +
    m ((b Sin[θns[t]] Sin[ϕns[t]] θns′[t] - (a + b) Sin[θs[t]] Sin[ϕs[t]] θs′[t] -
          b Cos[θns[t]] Cos[ϕns[t]] ϕns′[t] + (a + b) Cos[θs[t]] Cos[ϕs[t]] ϕs′[t])^2 +
       (b Cos[θns[t]] Sin[ϕns[t]] θns′[t] - (a + b) Cos[θs[t]] Sin[ϕs[t]] θs′[t] + b Cos[ϕns[t]] Sin[θns[t]] ϕns′[t] -
          (a + b) Cos[ϕs[t]] Sin[θs[t]] ϕs′[t])^2 + (b Sin[ϕns[t]] ϕns′[t] - (a + b) Sin[ϕs[t]] ϕs′[t])^2))
```

- **The following shows that this Lagrangian accurately simplifies to the previously derived 2D model by constraining this model to two dimensions:**

```
FullSimplify[L /. {θs[t] → Pi/2, θns[t] → Pi/2, θs'[t] → 0, θns'[t] → 0}]
```

$$\frac{1}{2}\left(2\,b\,g\,m\,\mathrm{Cos}[\phi ns[t]] - 2\,g\,((2\,a+b)\,m+(a+b)\,M)\,\mathrm{Cos}[\phi s[t]] + b^2\,m\,\phi ns'[t]^2 - 2\,b\,(a+b)\,m\,\mathrm{Cos}[\phi ns[t]-\phi s[t]]\,\phi ns'[t]\,\phi s'[t] + ((2\,a^2+2\,a\,b+b^2)\,m+(a+b)^2\,M)\,\phi s'[t]^2\right)$$

Equations of Motion

```
Ldθns = D[L, θns[t]]; Ldφns = D[L, φns[t]]; Ldθs = D[L, θs[t]]; Ldφs = D[L, φs[t]]; Ldθnsdot = D[L, θns'[t]];
Ldθnsdotdt = D[Ldθnsdot, t]; Ldφnsdot = D[L, φns'[t]]; Ldφnsdotdt = D[Ldφnsdot, t];
Ldθsdot = D[L, θs'[t]]; Ldθsdotdt = D[Ldθsdot, t]; Ldφsdot = D[L, φs'[t]]; Ldφsdotdt = D[Ldφsdot, t];
```

```
FullSimplify[Ldφnsdotdt - Ldφns == 0]
FullSimplify[Ldφsdotdt - Ldφs == 0]
FullSimplify[Ldθnsdotdt - Ldθns == 0]
FullSimplify[Ldθsdotdt - Ldθs == 0]
```

$$b\,m\,\big(b\,\mathrm{Sin}[2\,\phi ns[t]]\,\theta ns'[t]^2 - 2\,(g\,\mathrm{Sin}[\phi ns[t]] + b\,\phi ns''[t]) + 2\,(a+b)\,\mathrm{Cos}[\phi s[t]]\,(2\,\mathrm{Cos}[\phi ns[t]]\,\mathrm{Sin}[\theta ns[t]-\theta s[t]]\,\theta s'[t]\,\phi s'[t] + \mathrm{Sin}[\phi ns[t]]\,\phi s'[t]^2 + \mathrm{Cos}[\theta ns[t]-\theta s[t]]\,\mathrm{Cos}[\phi ns[t]]\,\phi s''[t]) + 2\,(a+b)\,\mathrm{Sin}[\phi s[t]]\,(\mathrm{Cos}[\phi ns[t]]\,(-\mathrm{Cos}[\theta ns[t]-\theta s[t]]\,(\theta s'[t]^2+\phi s'[t]^2) + \mathrm{Sin}[\theta ns[t]-\theta s[t]]\,\theta s''[t]) + \mathrm{Sin}[\phi ns[t]]\,\phi s''[t])\big) == 0$$

$$2\,g\,(b\,(m+M)+a\,(2\,m+M))\,\mathrm{Sin}[\phi s[t]] + 2\,b\,(a+b)\,m\,\mathrm{Sin}[\phi ns[t]]\,(\mathrm{Cos}[\phi s[t]]\,(-\mathrm{Cos}[\theta ns[t]-\theta s[t]]\,(\theta ns'[t]^2+\phi ns'[t]^2) - \mathrm{Sin}[\theta ns[t]-\theta s[t]]\,\theta ns''[t]) + \mathrm{Sin}[\phi s[t]]\,\phi ns''[t]) + 2\,b\,(a+b)\,m\,\mathrm{Cos}[\phi ns[t]]\,(\mathrm{Sin}[\phi s[t]]\,\phi ns'[t]^2 + \mathrm{Cos}[\phi s[t]]\,(-2\,\mathrm{Sin}[\theta ns[t]-\theta s[t]]\,\theta ns'[t]\,\phi ns'[t] + \mathrm{Cos}[\theta ns[t]-\theta s[t]]\,\phi ns''[t])) + ((2\,a^2+2\,a\,b+b^2)\,m+(a+b)^2\,M)\,(\mathrm{Sin}[2\,\phi s[t]]\,\theta s'[t]^2 - 2\,\phi s''[t]) == 0$$

$$b\,m\,\mathrm{Sin}[\phi ns[t]]\,\big(b\,(2\,\mathrm{Cos}[\phi ns[t]]\,\theta ns'[t]\,\phi ns'[t] + \mathrm{Sin}[\phi ns[t]]\,\theta ns''[t]) + (a+b)\,\mathrm{Sin}[\phi s[t]]\,(-\mathrm{Sin}[\theta ns[t]-\theta s[t]]\,(\theta s'[t]^2+\phi s'[t]^2) - \mathrm{Cos}[\theta ns[t]-\theta s[t]]\,\theta s''[t]) - (a+b)\,\mathrm{Cos}[\phi s[t]]\,(2\,\mathrm{Cos}[\theta ns[t]-\theta s[t]]\,\theta s'[t]\,\phi s'[t] - \mathrm{Sin}[\theta ns[t]-\theta s[t]]\,\phi s''[t])\big) == 0$$

$$\mathrm{Sin}[\phi s[t]]\,\big(b\,(a+b)\,m\,\mathrm{Sin}[\phi ns[t]]\,(\mathrm{Sin}[\theta ns[t]-\theta s[t]]\,(\theta ns'[t]^2+\phi ns'[t]^2) - \mathrm{Cos}[\theta ns[t]-\theta s[t]]\,\theta ns''[t]) + ((2\,a^2+2\,a\,b+b^2)\,m+(a+b)^2\,M)\,(2\,\mathrm{Cos}[\phi s[t]]\,\theta s'[t]\,\phi s'[t] + \mathrm{Sin}[\phi s[t]]\,\theta s''[t]) - b\,(a+b)\,m\,\mathrm{Cos}[\phi ns[t]]\,(2\,\mathrm{Cos}[\theta ns[t]-\theta s[t]]\,\theta ns'[t]\,\phi ns'[t] + \mathrm{Sin}[\theta ns[t]-\theta s[t]]\,\phi ns''[t])\big) == 0$$

■ Generalized model

$$M[\theta]\,\theta'' + F[\theta, \theta']\,\theta' + q[\theta] = 0$$

where, $\theta = \begin{pmatrix} \phi ns[t] \\ \phi s[t] \\ \theta ns[t] \\ \theta s[t] \end{pmatrix}$

```
M[φns[t], φs[t], θns[t], θs[t]] :=
 {{b^2 m, -b (a + b) m (Cos[θns[t] - θs[t]] Cos[φns[t]] Cos[φs[t]] + Sin[φns[t]] Sin[φs[t]]),
   0, -b (a + b) m Cos[φns[t]] Sin[θns[t] - θs[t]] Sin[φs[t]]},
  {-b (a + b) m (Cos[θns[t] - θs[t]] Cos[φns[t]] Cos[φs[t]] + Sin[φns[t]] Sin[φs[t]]),
   ((2 a^2 + 2 a b + b^2) m + (a + b)^2 M), b (a + b) m Cos[φs[t]] Sin[θns[t] - θs[t]] Sin[φns[t]], 0},
  {0, b (a + b) m Cos[φs[t]] Sin[θns[t] - θs[t]] Sin[φns[t]], b^2 m Sin[φns[t]]^2,
   -b (a + b) m Cos[θns[t] - θs[t]] Sin[φns[t]] Sin[φs[t]]},
  {-b (a + b) m Cos[φns[t]] Sin[θns[t] - θs[t]] Sin[φs[t]], 0, -b (a + b) m Cos[θns[t] - θs[t]] Sin[φns[t]] Sin[φs[t]],
   ((2 a^2 + 2 a b + b^2) m + (a + b)^2 M) Sin[φs[t]]^2}}

F[φns[t], φs[t], θns[t], θs[t], φns'[t], φs'[t], θns'[t], θs'[t]] :=
 {{0, b (a + b) m (Cos[θns[t] - θs[t]] Cos[φns[t]] Sin[φs[t]] φs'[t] -
      Cos[φs[t]] (Cos[φns[t]] Sin[θns[t] - θs[t]] θs'[t] + Sin[φns[t]] φs'[t])), -1/2 b^2 m Sin[2 φns[t]] θns'[t],
   b (a + b) m Cos[φns[t]] (Cos[θns[t] - θs[t]] Sin[φs[t]] θs'[t] - Cos[φs[t]] Sin[θns[t] - θs[t]] φs'[t])},
  {b (a + b) m (Cos[φns[t]] Cos[φs[t]] Sin[θns[t] - θs[t]] θns'[t] +
      (Cos[θns[t] - θs[t]] Cos[φs[t]] Sin[φns[t]] - Cos[φns[t]] Sin[φs[t]]) φns'[t]), 0,
   b (a + b) m Cos[φs[t]] (Cos[θns[t] - θs[t]] Sin[φns[t]] θns'[t] + Cos[φns[t]] Sin[θns[t] - θs[t]] φns'[t]),
   -1/2 ((2 a^2 + 2 a b + b^2) m + (a + b)^2 M) Sin[2 φs[t]] θs'[t]},
  {b^2 m Cos[φns[t]] Sin[φns[t]] θns'[t], -b (a + b) m Sin[φns[t]]
    (Cos[θns[t] - θs[t]] Cos[φs[t]] θs'[t] + Sin[θns[t] - θs[t]] Sin[φs[t]] φs'[t]), b^2 m Cos[φns[t]] Sin[φns[t]] φns'[t],
   -b (a + b) m Sin[φns[t]] (Sin[θns[t] - θs[t]] Sin[φs[t]] θs'[t] + Cos[θns[t] - θs[t]] Cos[φs[t]] φs'[t])},
  {-b (a + b) m Sin[φs[t]] (Cos[θns[t] - θs[t]] Cos[φns[t]] θns'[t] - Sin[θns[t] - θs[t]] Sin[φns[t]] φns'[t]),
   ((2 a^2 + 2 a b + b^2) m + (a + b)^2 M) Cos[φs[t]] Sin[φs[t]] θs'[t],
   -b (a + b) m Sin[φs[t]] (-Sin[θns[t] - θs[t]] Sin[φns[t]] θns'[t] + Cos[θns[t] - θs[t]] Cos[φns[t]] φns'[t]),
   ((2 a^2 + 2 a b + b^2) m + (a + b)^2 M) Cos[φs[t]] Sin[φs[t]] φs'[t]}}

g[φns[t], φs[t], θns[t], θs[t]] := {b g m Sin[φns[t]], -g (b (m + M) + a (2 m + M)) Sin[φs[t]], 0, 0}
```

■ Normalized form (μ = M/m, β = b/a)

$$m\,a^2\left(M[\theta]\,\theta'' + F[\theta,\theta']\,\theta' + \frac{1}{a}\,q[\theta]\right) = 0$$

$$\text{where, } \theta = \begin{pmatrix} \phi ns[t] \\ \phi s[t] \\ \theta ns[t] \\ \theta s[t] \end{pmatrix}$$

```
M11 := β^2;
M12 := -(β + 1) β (Cos[θns[t] - θs[t]] Cos[ϕns[t]] Cos[ϕs[t]] + Sin[ϕns[t]] Sin[ϕs[t]]);
M13 := 0;
M14 := -(β + 1) β Cos[ϕns[t]] Sin[θns[t] - θs[t]] Sin[ϕs[t]];
M21 := -(β + 1) β (Cos[θns[t] - θs[t]] Cos[ϕns[t]] Cos[ϕs[t]] + Sin[ϕns[t]] Sin[ϕs[t]]);
M22 := ((1 + β)^2 (μ + 1) + 1);
M23 := (β + 1) β Cos[ϕs[t]] Sin[θns[t] - θs[t]] Sin[ϕns[t]];
M24 := 0;  M31 = 0;
M32 := (β + 1) β Cos[ϕs[t]] Sin[θns[t] - θs[t]] Sin[ϕns[t]];
M33 := β^2 Sin[ϕns[t]]^2;
M34 := -(β + 1) β Cos[θns[t] - θs[t]] Sin[ϕns[t]] Sin[ϕs[t]];
M41 := -(β + 1) β Cos[ϕns[t]] Sin[θns[t] - θs[t]] Sin[ϕs[t]];
M42 := 0;
M43 := -(β + 1) β Cos[θns[t] - θs[t]] Sin[ϕns[t]] Sin[ϕs[t]];
M44 := ((1 + β)^2 (μ + 1) + 1) Sin[ϕs[t]]^2;

F11 := 0;
F12 := (β + 1) β (Cos[θns[t] - θs[t]] Cos[ϕns[t]] Sin[ϕs[t]] ϕs'[t] -
    Cos[ϕs[t]] (Cos[ϕns[t]] Sin[θns[t] - θs[t]] θs'[t] + Sin[ϕns[t]] ϕs'[t]));
F13 := -1/2 β^2 Sin[2 ϕns[t]] θns'[t];
F14 := (β + 1) β Cos[ϕns[t]] (Cos[θns[t] - θs[t]] Sin[ϕs[t]] θs'[t] - Cos[ϕs[t]] Sin[θns[t] - θs[t]] ϕs'[t]);
F21 := ((β + 1) β (Cos[ϕns[t]] Cos[ϕs[t]] Sin[θns[t] - θs[t]] θns'[t] +
     (Cos[θns[t] - θs[t]] Cos[ϕs[t]] Sin[ϕns[t]] - Cos[ϕns[t]] Sin[ϕs[t]]) ϕns'[t]));
F22 := 0;
F23 := (β + 1) β Cos[ϕs[t]] (Cos[θns[t] - θs[t]] Sin[ϕns[t]] θns'[t] + Cos[ϕns[t]] Sin[θns[t] - θs[t]] ϕns'[t]);
F24 := -1/2 ((1 + β)^2 (μ + 1) + 1) Sin[2 ϕs[t]] θs'[t];
```

```
F31 := β² Cos[ϕns[t]] Sin[ϕns[t]] θns′[t];
F32 := −(β + 1) β Sin[ϕns[t]] (Cos[θns[t] − θs[t]] Cos[ϕs[t]] θs′[t] + Sin[θns[t] − θs[t]] Sin[ϕs[t]] ϕs′[t]);
F33 := β² Cos[ϕns[t]] Sin[ϕns[t]] ϕns′[t];
F34 := −(β + 1) β Sin[ϕns[t]] (Sin[θns[t] − θs[t]] Sin[ϕs[t]] θs′[t] + Cos[θns[t] − θs[t]] Cos[ϕs[t]] ϕs′[t]);
F41 := −(β + 1) β Sin[ϕs[t]] (Cos[θns[t] − θs[t]] Cos[ϕns[t]] θns′[t] − Sin[θns[t] − θs[t]] Sin[ϕns[t]] ϕns′[t]);
F42 := ((1 + β)² (μ + 1) + 1) Cos[ϕs[t]] Sin[ϕs[t]] θs′[t];
F43 := −(β + 1) β Sin[ϕs[t]] (−Sin[θns[t] − θs[t]] Sin[ϕns[t]] θns′[t] + Cos[θns[t] − θs[t]] Cos[ϕns[t]] ϕns′[t]);
F44 := ((1 + β)² (μ + 1) + 1) Cos[ϕs[t]] Sin[ϕs[t]] ϕs′[t];

M[ϕns[t], ϕs[t], θns[t], θs[t]] :=
  {{M11, M12, M13, M14}, {M21, M22, M23, M24}, {M31, M32, M33, M34}, {M41, M42, M43, M44}};
F[ϕns[t], ϕs[t], θns[t], θs[t], ϕns'[t], ϕs'[t], θns'[t], θs'[t]] :=
  {{F11, F12, F13, F14}, {F21, F22, F23, F24}, {F31, F32, F33, F34}, {F41, F42, F43, F44}};

q[ϕns[t], ϕs[t], θns[t], θs[t]] := {β g Sin[ϕns[t]], −g ((β + 1) (μ + 1) + 1) Sin[ϕs[t]], 0, 0};
```

Testing Correctness

■ Testing equivalence of the above matrix-form equations of motion:

```
Zeroed = FullSimplify[M[ϕns[t], ϕs[t], θns[t], θs[t]].{ϕns''[t], ϕs''[t], θns''[t], θs''[t]} +
    F[ϕns[t], ϕs[t], θns[t], θs[t], ϕns'[t], ϕs'[t], θns'[t], θs'[t]].{ϕns'[t], ϕs'[t], θns'[t], θs'[t]} +
    1/a q[ϕns[t], ϕs[t], θns[t], θs[t]] /. {μ → M / m, β → b / a}];
```

```
FullSimplify[m a^2 Zeroed ==
  {-1/2 b m (b Sin[2 ϕns[t]] θns′[t]^2 - 2 (g Sin[ϕns[t]] + b ϕns″[t]) + 2 (a + b) Cos[ϕs[t]] (2 Cos[ϕns[t]] Sin[θns[t] - θs[t]]
          θs′[t] ϕs′[t] + Sin[ϕns[t]] ϕs′[t]^2 + Cos[θns[t] - θs[t]] Cos[ϕns[t]] ϕs″[t]) + 2 (a + b) Sin[ϕs[t]]
        (Cos[ϕns[t]] (-Cos[θns[t] - θs[t]] (θs′[t]^2 + ϕs′[t]^2) + Sin[θns[t] - θs[t]] θs″[t]) + Sin[ϕns[t]] ϕs″[t])),
   -g (b (m + M) + a (2 m + M)) Sin[ϕs[t]] - b (a + b) m Sin[ϕns[t]]
      (Cos[ϕs[t]] (-Cos[θns[t] - θs[t]] (θns′[t]^2 + ϕns′[t]^2) - Sin[θns[t] - θs[t]] θns″[t]) + Sin[ϕs[t]] ϕns″[t]) - b
      (a + b) m Cos[ϕns[t]]
      (Sin[ϕs[t]] ϕns′[t]^2 + Cos[ϕs[t]] (-2 Sin[θns[t] - θs[t]] θns′[t] ϕns′[t] + Cos[θns[t] - θs[t]] ϕns″[t])) -
     1/2 ((2 a^2 + 2 a b + b^2) m + (a + b)^2 M) (Sin[2 ϕs[t]] θs′[t]^2 - 2 ϕs″[t]),
   b m Sin[ϕns[t]] (b (2 Cos[ϕns[t]] θns′[t] ϕns′[t] + Sin[ϕns[t]] θns″[t]) +
       (a + b) Sin[ϕs[t]] (-Sin[θns[t] - θs[t]] (θs′[t]^2 + ϕs′[t]^2) - Cos[θns[t] - θs[t]] θs″[t]) -
       (a + b) Cos[ϕs[t]] (2 Cos[θns[t] - θs[t]] θs′[t] ϕs′[t] - Sin[θns[t] - θs[t]] ϕs″[t])),
   Sin[ϕs[t]] (b (a + b) m Sin[ϕns[t]] (Sin[θns[t] - θs[t]] (θns′[t]^2 + ϕns′[t]^2) - Cos[θns[t] - θs[t]] θns″[t]) +
       ((2 a^2 + 2 a b + b^2) m + (a + b)^2 M) (2 Cos[ϕs[t]] θs′[t] ϕs′[t] + Sin[ϕs[t]] θs″[t]) -
       b (a + b) m Cos[ϕns[t]] (2 Cos[θns[t] - θs[t]] θns′[t] ϕns′[t] + Sin[θns[t] - θs[t]] ϕns″[t]))}]
```

```
True
```

▪ Testing correctness by fixing model in two dimensions, setting θns[t] → Pi/2, θs[t] → Pi/2:

```
MatrixForm[FullSimplify[M[ϕns[t], ϕs[t], θns[t], θs[t]] /. {θns[t] → Pi / 2, θs[t] → Pi / 2}]]
```

β^2	-β (1 + β) Cos[ϕns[t] - ϕs[t]]	0	0
-β (1 + β) Cos[ϕns[t] - ϕs[t]]	1 + (1 + β)^2 (1 + μ)	0	0
0	0	β^2 Sin[ϕns[t]]^2	-β (1 + β) Sin[ϕns[t]] Sin
0	0	-β (1 + β) Sin[ϕns[t]] Sin[ϕs[t]]	(1 + (1 + β)^2 (1 + μ)) Sin[ϕ

We see in the above simplified matrix that the top-left 2x2 submatrix is exactly the same as Goswami's. Only this 2x2 matrix matters because M is multiplied by the second order angle vector, and the θ terms are all zero.

```
MatrixForm[FullSimplify[F[ϕns[t], ϕs[t], θns[t], θs[t], ϕns'[t], ϕs'[t], θns'[t], θs'[t]] /.
   {θns[t] → Pi/2, θs[t] → Pi/2, θns'[t] → 0, θs'[t] → 0}]]
```

$$\begin{pmatrix} 0 & -\beta\,(1+\beta)\,\text{Sin}[\phi ns[t]-\phi s[t]]\,\phi s'[t] & 0 & 0 \\ \beta\,(1+\beta)\,\text{Sin}[\phi ns[t]-\phi s[t]]\,\phi ns'[t] & 0 & 0 & 0 \\ 0 & 0 & \beta^2\,\text{Cos}[\phi ns[t]]\,\text{Sin}[\phi ns[t]]\,\phi ns'[t] & -\beta\,(1 \\ 0 & 0 & -\beta\,(1+\beta)\,\text{Cos}[\phi ns[t]]\,\text{Sin}[\phi s[t]]\,\phi ns'[t] & (1+ \end{pmatrix}$$

Similarly, the above simplified top-left submatrix is the same as Goswami's.

```
MatrixForm[q[ϕns[t], ϕs[t], θns[t], θs[t]]]
```

$$\begin{pmatrix} g\,\beta\,\text{Sin}[\phi ns[t]] \\ -g\,(1+(1+\beta)\,(1+\mu))\,\text{Sin}[\phi s[t]] \\ 0 \\ 0 \end{pmatrix}$$

Clearly, the potential remains the same since it is independent of the θ terms. Thus, all the matrices simplify to the 2D model when fixed in two dimensions.

Simplification investigations

■ Fix the angle (2 α) between the legs:

```
FullSimplify[L /. {ϕs[t] → 2 α + ϕns[t], ϕs'[t] → ϕns'[t]}]
```

$$\frac{1}{2}\,(2\,b\,g\,m\,\text{Cos}[\phi ns[t]] - 2\,g\,((2\,a+b)\,m+(a+b)\,M)\,\text{Cos}[2\,\alpha+\phi ns[t]] + b^2\,m\,\text{Sin}[\phi ns[t]]^2\,\theta ns'[t]^2 + ((2\,a^2+2\,a\,b+b^2)\,m+(a+b)^2\,M)\,\text{Sin}[2\,\alpha+\phi ns[t]]^2\,\theta s'[t]^2 - 2\,b\,(a+b)\,m\,\text{Cos}[\phi ns[t]]\,\text{Sin}[\theta ns[t]-\theta s[t]]\,\text{Sin}[2\,\alpha+\phi ns[t]]\,\theta s'[t]\,\phi ns'[t] + (2\,(a^2+a\,b+b^2)\,m+(a+b)^2\,M - 2\,b\,(a+b)\,m\,(\text{Cos}[\theta ns[t]-\theta s[t]]\,\text{Cos}[\phi ns[t]]\,\text{Cos}[2\,\alpha+\phi ns[t]]+\text{Sin}[\phi ns[t]]\,\text{Sin}[2\,\alpha+\phi ns[t]]))\,\phi ns'[t]^2 + 2\,b\,(a+b)\,m\,\text{Sin}[\phi ns[t]]\,\theta ns'[t]\,(-\text{Cos}[\theta ns[t]-\theta s[t]]\,\text{Sin}[2\,\alpha+\phi ns[t]]\,\theta s'[t]+\text{Cos}[2\,\alpha+\phi ns[t]]\,\text{Sin}[\theta ns[t]-\theta s[t]]\,\phi ns'[t]))$$

This results in one cyclic variable, ϕs[t]; L is in terms of ϕns[t], θns[t] , θs[t], ϕns'[t], θns'[t] , θs'[t].

■ Take the limit as $\frac{M}{m}$ approaches infinity :

```
MatrixForm[FullSimplify[{M[ϕns[t], ϕs[t], θns[t] , θs[t]][[1]], Expand[M[ϕns[t], ϕs[t], θns[t] , θs[t]][[2]] / μ],
   M[ϕns[t], ϕs[t], θns[t] , θs[t]][[3]], Expand[M[ϕns[t], ϕs[t], θns[t] , θs[t]][[4]] / μ]} /. μ → ∞]]
```

```
( β^2   -β (1 + β) (Cos[θns[t] - θs[t]] Cos[ϕns[t]] Cos[ϕs[t]] + Sin[ϕns[t]] Sin[ϕs[t]])   0                  -β (1 + β) Cos[ϕns[
  0     (1 + β)^2                                                                            0                  0
  0     β (1 + β) Cos[ϕs[t]] Sin[θns[t] - θs[t]] Sin[ϕns[t]]                                 β^2 Sin[ϕns[t]]^2  -β (1 + β) Cos[θns[
  0     0                                                                                    0                  (1 + β)^2 Sin[ϕs[t]]
```

This results in no cyclic variables, so the above approach does not offer a significant simplification of the model.

■ Take the limit as $\frac{b}{a}$ approaches infinity :

```
MatrixForm[FullSimplify[Expand[(M[ϕns[t], ϕs[t], θns[t] , θs[t]] / β^2)] /. β → ∞]]
```

```
( 1                                                                      -Cos[θns[t] - θs[t]] Cos[ϕns[t]] Cos[ϕs[t]] - Sin[
  -Cos[θns[t] - θs[t]] Cos[ϕns[t]] Cos[ϕs[t]] - Sin[ϕns[t]] Sin[ϕs[t]]   1 + μ
  0                                                                      Cos[ϕs[t]] Sin[θns[t] - θs[t]] Sin[ϕns[t]]
  -Cos[ϕns[t]] Sin[θns[t] - θs[t]] Sin[ϕs[t]]                            0
```

This results in no cyclic variables, so the above approach does not offer a significant simplification of the model

■ Set θns[t] equal to θs[t] (the x-y plane angles are equal):

```
Msimple[ϕns[t], ϕs[t], θns[t] , θs[t]] = FullSimplify[M[ϕns[t], ϕs[t], θns[t] , θs[t]] /. {θns[t] - θs[t] → 0}];
MatrixForm[Msimple[ϕns[t], ϕs[t], θns[t] , θs[t]]]
```

```
( β^2                             -β (1 + β) Cos[ϕns[t] - ϕs[t]]   0                                  0
  -β (1 + β) Cos[ϕns[t] - ϕs[t]]  1 + (1 + β)^2 (1 + μ)            0                                  0
  0                               0                                β^2 Sin[ϕns[t]]^2                  -β (1 + β) Sin[ϕns[t]] Sin
  0                               0                                -β (1 + β) Sin[ϕns[t]] Sin[ϕs[t]]  (1 + (1 + β)^2 (1 + μ)) Sin[ϕ
```

```
MatrixForm[q[ϕns[t], ϕs[t], θns[t] , θs[t]]]
```

$$\begin{pmatrix} g\,\beta\,\text{Sin}[\phi ns[t]] \\ -g\,(1+(1+\beta)\,(1+\mu))\,\text{Sin}[\phi s[t]] \\ 0 \\ 0 \end{pmatrix}$$

This results in two cyclic variables, θns[t] and θs[t].

Using the formula (remember that the generalized system is normalized, so include all terms):

$$L\,(q,\,q') = \frac{1}{2}\,q'^{T}.(m\,a^2\,M\,(q)).q' - U\,(q) = \frac{1}{2}\,m\,a^2\,(q'^{T}.\,M\,(q).q') - U\,(q)$$

where g(q) = gradient(U(q)), thus

$$U\,(q) = \text{Integrate}[-m\,a\,g\,(1+(1+\beta)\,(1+\mu))\,\text{Sin}[\phi s[t]],\,\phi s[t]] + \text{Integrate}[m\,a\,g\,\beta\,\text{Sin}[\phi ns[t]],\,\phi ns[t]]$$

... we get the following Lagrangian in terms of ϕns[t], ϕs[t], ϕns'[t], ϕs'[t], θs'[t]:

```
Lsimple[ϕns[t], ϕs[t], ϕns'[t], ϕs'[t], θs'[t]] =
 FullSimplify[1/2 m a^2 ({ϕns'[t], ϕs'[t], θns'[t] , θs'[t]}. Msimple[ϕns[t], ϕs[t], θns[t] , θs[t]].
     {ϕns'[t], ϕs'[t], θns'[t] , θs'[t]}) - U /. {μ → M / m, β → b / a, θns'[t] → θs'[t]}]
```

$$b\,g\,m\,\text{Cos}[\phi ns[t]] - g\,(b\,(m+M) + a\,(2\,m+M))\,\text{Cos}[\phi s[t]] + \frac{1}{2}\,((b^2\,m\,\text{Sin}[\phi ns[t]]^2 - 2\,b\,(a+b)\,m\,\text{Sin}[\phi ns[t]]\,\text{Sin}[\phi s[t]] + ((2\,a^2+2\,a\,b+b^2)\,m + (a+b)^2\,M)\,\text{Sin}[\phi s[t]]^2)\,\theta s'[t]^2 + b^2\,m\,\phi ns'[t]^2 - 2\,b\,(a+b)\,m\,\text{Cos}[\phi ns[t] - \phi s[t]]\,\phi ns'[t]\,\phi s'[t] + ((2\,a^2+2\,a\,b+b^2)\,m + (a+b)^2\,M)\,\phi s'[t]^2)$$

Reduced 3D Model

■ Routhian Formulation

$$\text{Routhian} = L[\phi,\,\phi',\,\Theta'] - c\,\Theta'$$

$$\text{where } \phi = \begin{pmatrix} \phi ns[t] \\ \phi s[t] \end{pmatrix},\ \Theta' = c\,/\,m2[\phi]$$

```
M1[ϕns[t], ϕs[t]] :=
 {{b^2 m, -b (a + b) m Cos[ϕns[t] - ϕs[t]]}, {-b (a + b) m Cos[ϕns[t] - ϕs[t]], (2 a^2 + 2 a b + b^2) m + (a + b)^2 M}}
M2[ϕns[t], ϕs[t]] := {{b^2 m Sin[ϕns[t]]^2, -b (a + b) m Sin[ϕns[t]] Sin[ϕs[t]]},
  {-b (a + b) m Sin[ϕns[t]] Sin[ϕs[t]], ((2 a^2 + 2 a b + b^2) m + (a + b)^2 M) Sin[ϕs[t]]^2}}
```

```
ϕ[t] = {{ϕns[t]}, {ϕs[t]}};
θ[t] = {{θns[t]}, {θs[t]}};
m2[ϕns[t], ϕs[t]] =
  FullSimplify[First[First[Transpose[θ[t]].M2[ϕns[t], ϕs[t]].θ[t] /. {θns[t] → θs[t]}]] / θs[t]^2];
θdot = c / m2[ϕns[t], ϕs[t]]
```

$$\frac{c}{b^2 m \,\mathrm{Sin}[\phi ns[t]]^2 - 2 b (a+b) m \,\mathrm{Sin}[\phi ns[t]]\,\mathrm{Sin}[\phi s[t]] + ((2a^2 + 2ab + b^2) m + (a+b)^2 M)\,\mathrm{Sin}[\phi s[t]]^2}$$

```
R = FullSimplify[Lsimple[ϕns[t], ϕs[t], ϕns'[t], ϕs'[t], θs'[t]] - c θs'[t] /. θs'[t] → θdot]
```

$$\frac{1}{2}\Bigg(2 b g m \,\mathrm{Cos}[\phi ns[t]] - 2 g ((2a+b) m + (a+b) M)\,\mathrm{Cos}[\phi s[t]] - \frac{c^2}{b^2 m \,\mathrm{Sin}[\phi ns[t]]^2 - 2 b (a+b) m \,\mathrm{Sin}[\phi ns[t]]\,\mathrm{Sin}[\phi s[t]] + ((2a^2+2ab+b^2) m + (a+b)^2 M)\,\mathrm{Sin}[\phi s[t]]^2} + b^2 m\, \phi ns'[t]^2 - 2 b (a+b) m \,\mathrm{Cos}[\phi ns[t] - \phi s[t]]\, \phi ns'[t]\, \phi s'[t] + ((2a^2+2ab+b^2) m + (a+b)^2 M)\, \phi s'[t]^2\Bigg)$$

■ Equations of Motion

```
Rdϕns = D[R, ϕns[t]]; Rdϕs = D[R, ϕs[t]]; Rdϕnsdot = D[R, ϕns'[t]];
Rdϕnsdotdt = D[Rdϕnsdot, t]; Rdϕsdot = D[R, ϕs'[t]]; Rdϕsdotdt = D[Rdϕsdot, t];
FullSimplify[Rdϕnsdotdt - Rdϕns == 0]
FullSimplify[Rdϕsdotdt - Rdϕs == 0]
```

$$b m \Bigg(g \,\mathrm{Sin}[\phi ns[t]] + \frac{c^2 \,\mathrm{Cos}[\phi ns[t]]\,(-b \,\mathrm{Sin}[\phi ns[t]] + (a+b)\,\mathrm{Sin}[\phi s[t]])}{(b^2 m \,\mathrm{Sin}[\phi ns[t]]^2 - 2 b (a+b) m \,\mathrm{Sin}[\phi ns[t]]\,\mathrm{Sin}[\phi s[t]] + ((2a^2+2ab+b^2) m + (a+b)^2 M)\,\mathrm{Sin}[\phi s[t]]^2)^2} - (a+b)\,\mathrm{Sin}[\phi ns[t] - \phi s[t]]\, \phi s'[t]^2 + b\, \phi ns''[t] - (a+b)\,\mathrm{Cos}[\phi ns[t] - \phi s[t]]\, \phi s''[t]\Bigg) == 0$$

$$b (a+b) m \,\mathrm{Sin}[\phi ns[t] - \phi s[t]]\, \phi ns'[t]^2 + ((2a^2+2ab+b^2) m + (a+b)^2 M)\, \phi s''[t] == g (b (m+M) + a (2m+M))\,\mathrm{Sin}[\phi s[t]] + \frac{c^2 \,\mathrm{Cos}[\phi s[t]]\,(-b (a+b) m \,\mathrm{Sin}[\phi ns[t]] + ((2a^2+2ab+b^2) m + (a+b)^2 M)\,\mathrm{Sin}[\phi s[t]])}{(b^2 m \,\mathrm{Sin}[\phi ns[t]]^2 - 2 b (a+b) m \,\mathrm{Sin}[\phi ns[t]]\,\mathrm{Sin}[\phi s[t]] + ((2a^2+2ab+b^2) m + (a+b)^2 M)\,\mathrm{Sin}[\phi s[t]]^2)^2} + b (a+b) m \,\mathrm{Cos}[\phi ns[t] - \phi s[t]]\, \phi ns''[t]$$

Generalized/Normalized Model (β = b/a, μ = M/m):

$$m\,a^2 \left(M[\phi]\,\phi'' + F[\phi,\phi']\,\phi' + \frac{1}{a}\,q[\phi] + \frac{1}{m^2 a^4}\,\text{aug}[\phi] \right) = 0$$

where, $\phi = \begin{pmatrix} \phi\text{ns}[t] \\ \phi\text{s}[t] \end{pmatrix}$

```
Mreduced[φns[t], φs[t]] := {{β^2, -β (1 + β) Cos[φns[t] - φs[t]]}, {-β (1 + β) Cos[φns[t] - φs[t]], 1 + (1 + β)^2 (1 + μ)}}
MatrixForm[Mreduced[φns[t], φs[t]]]
```

$$\begin{pmatrix} \beta^2 & -\beta\,(1+\beta)\,\text{Cos}[\phi\text{ns}[t]-\phi\text{s}[t]] \\ -\beta\,(1+\beta)\,\text{Cos}[\phi\text{ns}[t]-\phi\text{s}[t]] & 1+(1+\beta)^2\,(1+\mu) \end{pmatrix}$$

```
Freduced[φns[t], φs[t], φns'[t], φs'[t]] :=
 {{0, -β (1 + β) Sin[φns[t] - φs[t]] φs′[t]}, {β (1 + β) Sin[φns[t] - φs[t]] φns′[t], 0}}
MatrixForm[Freduced[φns[t], φs[t], φns'[t], φs'[t]]]
```

$$\begin{pmatrix} 0 & -\beta\,(1+\beta)\,\text{Sin}[\phi\text{ns}[t]-\phi\text{s}[t]]\,\phi\text{s}'[t] \\ \beta\,(1+\beta)\,\text{Sin}[\phi\text{ns}[t]-\phi\text{s}[t]]\,\phi\text{ns}'[t] & 0 \end{pmatrix}$$

```
qreduced[φns[t], φs[t]] := {g β Sin[φns[t]], -g (1 + (1 + β) (1 + μ)) Sin[φs[t]]}
MatrixForm[qreduced[φns[t], φs[t]]]
```

$$\begin{pmatrix} g\,\beta\,\text{Sin}[\phi\text{ns}[t]] \\ -g\,(1+(1+\beta)\,(1+\mu))\,\text{Sin}[\phi\text{s}[t]] \end{pmatrix}$$

$$\text{aug}[\phi\text{ns}[t],\ \phi\text{s}[t]] := \Big\{ \frac{c^2\,\beta\,\text{Cos}[\phi\text{ns}[t]]\,(-\beta\,\text{Sin}[\phi\text{ns}[t]] + (1+\beta)\,\text{Sin}[\phi\text{s}[t]])}{(\beta^2\,\text{Sin}[\phi\text{ns}[t]]^2 - 2\,(\beta+1)\,\beta\,\text{Sin}[\phi\text{ns}[t]]\,\text{Sin}[\phi\text{s}[t]] + ((1+\beta)^2\,(\mu+1)+1)\,\text{Sin}[\phi\text{s}[t]]^2)^2},$$
$$\frac{c^2\,\text{Cos}[\phi\text{s}[t]]\,((\beta+1)\,\beta\,\text{Sin}[\phi\text{ns}[t]] - ((1+\beta)^2\,(\mu+1)+1)\,\text{Sin}[\phi\text{s}[t]])}{(\beta^2\,\text{Sin}[\phi\text{ns}[t]]^2 - 2\,(\beta+1)\,\beta\,\text{Sin}[\phi\text{ns}[t]]\,\text{Sin}[\phi\text{s}[t]] + ((1+\beta)^2\,(\mu+1)+1)\,\text{Sin}[\phi\text{s}[t]]^2)^2} \Big\}$$

```
MatrixForm[aug[φns[t], φs[t]]]
```

$$\begin{pmatrix} \frac{c^2\,\beta\,\text{Cos}[\phi\text{ns}[t]]\,(-\beta\,\text{Sin}[\phi\text{ns}[t]]+(1+\beta)\,\text{Sin}[\phi\text{s}[t]])}{(\beta^2\,\text{Sin}[\phi\text{ns}[t]]^2-2\,\beta\,(1+\beta)\,\text{Sin}[\phi\text{ns}[t]]\,\text{Sin}[\phi\text{s}[t]]+(1+(1+\beta)^2\,(1+\mu))\,\text{Sin}[\phi\text{s}[t]]^2)^2} \\ \frac{c^2\,\text{Cos}[\phi\text{s}[t]]\,(\beta\,(1+\beta)\,\text{Sin}[\phi\text{ns}[t]]-(1+(1+\beta)^2\,(1+\mu))\,\text{Sin}[\phi\text{s}[t]])}{(\beta^2\,\text{Sin}[\phi\text{ns}[t]]^2-2\,\beta\,(1+\beta)\,\text{Sin}[\phi\text{ns}[t]]\,\text{Sin}[\phi\text{s}[t]]+(1+(1+\beta)^2\,(1+\mu))\,\text{Sin}[\phi\text{s}[t]]^2)^2} \end{pmatrix}$$

■ Testing equivalence of the above matrix-form equations of motion:

```
Zeroed = FullSimplify[
  Mreduced[ϕns[t], ϕs[t]].{ϕns''[t], ϕs''[t]} + Freduced[ϕns[t], ϕs[t], ϕns'[t], ϕs'[t]].{ϕns'[t], ϕs'[t]} +
   1/a qreduced[ϕns[t], ϕs[t]] + 1/(m^2 a^4) aug[ϕns[t], ϕs[t]] /. {μ → M / m, β → b / a}];
```

```
FullSimplify[m a^2 Zeroed == {b m (g Sin[ϕns[t]] +
      (c^2 Cos[ϕns[t]] (-b Sin[ϕns[t]] + (a + b) Sin[ϕs[t]])) /
      (b^2 m Sin[ϕns[t]]^2 - 2 b (a + b) m Sin[ϕns[t]] Sin[ϕs[t]] + ((2 a^2 + 2 a b + b^2) m + (a + b)^2 M) Sin[ϕs[t]]^2)^2 -
      (a + b) Sin[ϕns[t] - ϕs[t]] ϕs′[t]^2 + b ϕns′′[t] - (a + b) Cos[ϕns[t] - ϕs[t]] ϕs′′[t]),
   b (a + b) m Sin[ϕns[t] - ϕs[t]] ϕns′[t]^2 + ((2 a^2 + 2 a b + b^2) m + (a + b)^2 M) ϕs′′[t] - g (b (m + M) + a (2 m + M)) Sin[ϕs[t]] -
    (c^2 Cos[ϕs[t]] (-b (a + b) m Sin[ϕns[t]] + ((2 a^2 + 2 a b + b^2) m + (a + b)^2 M) Sin[ϕs[t]])) /
    (b^2 m Sin[ϕns[t]]^2 - 2 b (a + b) m Sin[ϕns[t]] Sin[ϕs[t]] + ((2 a^2 + 2 a b + b^2) m + (a + b)^2 M) Sin[ϕs[t]]^2)^2 -
    b (a + b) m Cos[ϕns[t] - ϕs[t]] ϕns′′[t]}]

True
```

■ ODE Form (x'[t] = f(x[t]))

```
MredInv[ϕns[t], ϕs[t]] = Inverse[Mreduced[ϕns[t], ϕs[t]]];
Simplify[-MredInv[ϕns[t], ϕs[t]].(Freduced[ϕns[t], ϕs[t], ϕns'[t], ϕs'[t]].{ϕns'[t], ϕs'[t]} +
    1/a qreduced[ϕns[t], ϕs[t]] + 1/(m^2 a^4) aug[ϕns[t], ϕs[t]])]
```

$$\Bigg\{-\Bigg(\beta\,(1+\beta)\,\mathrm{Cos}[\phi ns[t]-\phi s[t]]\,\Bigg(-\frac{g\,(2+\beta+\mu+\beta\,\mu)\,\mathrm{Sin}[\phi s[t]]}{a}+\frac{c^2\,\mathrm{Cos}[\phi s[t]]\,(\beta\,(1+\beta)\,\mathrm{Sin}[\phi ns[t]]-(1+(1+\beta)^2\,(1+\mu))\,\mathrm{Sin}[\phi s[t]])}{a^4\,m^2\,(\beta^2\,\mathrm{Sin}[\phi ns[t]]^2-2\,\beta\,(1+\beta)\,\mathrm{Sin}[\phi ns[t]]\,\mathrm{Sin}[\phi s[t]]+(1+(1+\beta)^2\,(1+\mu))\,\mathrm{Sin}[\phi s[t]]^2)^2}+\beta\,(1+\beta)\,\mathrm{Sin}[\phi ns[t]-\phi s[t]]\,\phi ns'[t]^2\Bigg)+\beta\,(1+(1+\beta)^2\,(1+\mu))\,\Bigg(\frac{g\,\mathrm{Sin}[\phi ns[t]]}{a}+\frac{c^2\,\mathrm{Cos}[\phi ns[t]]\,(-\beta\,\mathrm{Sin}[\phi ns[t]]+(1+\beta)\,\mathrm{Sin}[\phi s[t]])}{a^4\,m^2\,(\beta^2\,\mathrm{Sin}[\phi ns[t]]^2-2\,\beta\,(1+\beta)\,\mathrm{Sin}[\phi ns[t]]\,\mathrm{Sin}[\phi s[t]]+(1+(1+\beta)^2\,(1+\mu))\,\mathrm{Sin}[\phi s[t]]^2)^2}-(1+\beta)\,\mathrm{Sin}[\phi ns[t]-\phi s[t]]\,\phi s'[t]^2\Bigg)\Bigg)\Big/\left(\beta^2\,(1+(1+\beta)^2\,(1+\mu)-(1+\beta)^2\,\mathrm{Cos}[\phi ns[t]-\phi s[t]]^2)\right),\ -\Bigg(-\frac{g\,(2+\beta+\mu+\beta\,\mu)\,\mathrm{Sin}[\phi s[t]]}{a}+\frac{c^2\,\mathrm{Cos}[\phi s[t]]\,(\beta\,(1+\beta)\,\mathrm{Sin}[\phi ns[t]]-(1+(1+\beta)^2\,(1+\mu))\,\mathrm{Sin}[\phi s[t]])}{a^4\,m^2\,(\beta^2\,\mathrm{Sin}[\phi ns[t]]^2-2\,\beta\,(1+\beta)\,\mathrm{Sin}[\phi ns[t]]\,\mathrm{Sin}[\phi s[t]]+(1+(1+\beta)^2\,(1+\mu))\,\mathrm{Sin}[\phi s[t]]^2)^2}+\beta\,(1+\beta)\,\mathrm{Sin}[\phi ns[t]-\phi s[t]]\,\phi ns'[t]^2+(1+\beta)\,\mathrm{Cos}[\phi ns[t]-\phi s[t]]\,\Bigg(\frac{g\,\mathrm{Sin}[\phi ns[t]]}{a}+\frac{c^2\,\mathrm{Cos}[\phi ns[t]]\,(-\beta\,\mathrm{Sin}[\phi ns[t]]+(1+\beta)\,\mathrm{Sin}[\phi s[t]])}{a^4\,m^2\,(\beta^2\,\mathrm{Sin}[\phi ns[t]]^2-2\,\beta\,(1+\beta)\,\mathrm{Sin}[\phi ns[t]]\,\mathrm{Sin}[\phi s[t]]+(1+(1+\beta)^2\,(1+\mu))\,\mathrm{Sin}[\phi s[t]]^2)^2}-(1+\beta)\,\mathrm{Sin}[\phi ns[t]-\phi s[t]]\,\phi s'[t]^2\Bigg)\Bigg)\Big/\left(1+(1+\beta)^2\,(1+\mu)-(1+\beta)^2\,\mathrm{Cos}[\phi ns[t]-\phi s[t]]^2\right)\Bigg\}$$

- **Further simplifications:**

$$\mathbf{Simplify}\Big[\mathbf{FullSimplify}\Big[-\Big(\beta\,(1+\beta)\,\mathrm{Cos}[\phi ns[t]-\phi s[t]]\,\Big(-\frac{g\,(2+\beta+\mu+\beta\,\mu)\,\mathrm{Sin}[\phi s[t]]}{a}+\frac{c^2\,\mathrm{Cos}[\phi s[t]]\,(\beta\,(1+\beta)\,\mathrm{Sin}[\phi ns[t]]-(1+(1+\beta)^2\,(1+\mu))\,\mathrm{Sin}[\phi s[t]])}{a^4\,m^2\,(\beta^2\,\mathrm{Sin}[\phi ns[t]]^2-2\,\beta\,(1+\beta)\,\mathrm{Sin}[\phi ns[t]]\,\mathrm{Sin}[\phi s[t]]+(1+(1+\beta)^2\,(1+\mu))\,\mathrm{Sin}[\phi s[t]]^2)^2}+\beta\,(1+\beta)\,\mathrm{Sin}[\phi ns[t]-\phi s[t]]\,\phi ns'[t]^2\Big)+\beta\,(1+(1+\beta)^2\,(1+\mu))\,\Big(\frac{g\,\mathrm{Sin}[\phi ns[t]]}{a}+\frac{c^2\,\mathrm{Cos}[\phi ns[t]]\,(-\beta\,\mathrm{Sin}[\phi ns[t]]+(1+\beta)\,\mathrm{Sin}[\phi s[t]])}{a^4\,m^2\,(\beta^2\,\mathrm{Sin}[\phi ns[t]]^2-2\,\beta\,(1+\beta)\,\mathrm{Sin}[\phi ns[t]]\,\mathrm{Sin}[\phi s[t]]+(1+(1+\beta)^2\,(1+\mu))\,\mathrm{Sin}[\phi s[t]]^2)^2}-(1+\beta)\,\mathrm{Sin}[\phi ns[t]-\phi s[t]]\,\phi s'[t]^2\Big)\Big)\Big]\Big/\left(\beta^2\,(1+(1+\beta)^2\,(1+\mu))-(1+\beta)^2\,\mathrm{Cos}[\phi ns[t]-\phi s[t]]^2\right)\Big]$$

$$\Bigg(\frac{1}{a^4}\Big(\beta\,\Big(-a^3\,g\,(2+\mu+\beta\,(2+\beta)\,(1+\mu))\,\mathrm{Sin}[\phi ns[t]]+a^3\,g\,(1+\beta)\,(2+\beta+\mu+\beta\,\mu)\,\mathrm{Cos}[\phi ns[t]-\phi s[t]]\,\mathrm{Sin}[\phi s[t]]+\Big(c^2\,\Big((1+\beta)\,(2+\mu+\beta\,(2+\beta)\,(1+\mu))\,\mathrm{Sin}[\phi ns[t]-\phi s[t]]\,\mathrm{Sin}[\phi s[t]]^2+\frac{1}{4}\,\beta\,((3+2\,\mu+\beta\,(2+\beta)\,(1+2\,\mu)-(1+\beta)^2\,\mathrm{Cos}[2\,\phi s[t]])\,\mathrm{Sin}[2\,\phi ns[t]]-2\,(1+\beta)^2\,\mathrm{Sin}[\phi ns[t]]^2\,\mathrm{Sin}[2\,\phi s[t]])\Big)\Big)\Big/\left(m^2\,(\beta^2\,\mathrm{Sin}[\phi ns[t]]^2-2\,\beta\,(1+\beta)\,\mathrm{Sin}[\phi ns[t]]\,\mathrm{Sin}[\phi s[t]]+(2+\mu+\beta\,(2+\beta)\,(1+\mu))\,\mathrm{Sin}[\phi s[t]]^2)^2\right)\Big)\Big)+\beta\,(1+\beta)\,\mathrm{Sin}[\phi ns[t]-\phi s[t]]\,(-\beta\,(1+\beta)\,\mathrm{Cos}[\phi ns[t]-\phi s[t]]\,\phi ns'[t]^2+(2+\mu+\beta\,(2+\beta)\,(1+\mu))\,\phi s'[t]^2)\Bigg)\Big/\left(\beta^2\,(1+(1+\beta)^2\,(1+\mu))-(1+\beta)^2\,\mathrm{Cos}[\phi ns[t]-\phi s[t]]^2\right)$$

```
Simplify[Expand[-(-g (2 + β + μ + β μ) Sin[φs[t]]/a +
    c^2 Cos[φs[t]] (β (1 + β) Sin[φns[t]] - (1 + (1 + β)^2 (1 + μ)) Sin[φs[t]]) /
     (a^4 m^2 (β^2 Sin[φns[t]]^2 - 2 β (1 + β) Sin[φns[t]] Sin[φs[t]] + (1 + (1 + β)^2 (1 + μ)) Sin[φs[t]]^2)^2) +
    β (1 + β) Sin[φns[t] - φs[t]] φns'[t]^2 + (1 + β) Cos[φns[t] - φs[t]]
     (g Sin[φns[t]]/a + c^2 Cos[φns[t]] (-β Sin[φns[t]] + (1 + β) Sin[φs[t]]) /
       (a^4 m^2 (β^2 Sin[φns[t]]^2 - 2 β (1 + β) Sin[φns[t]] Sin[φs[t]] + (1 + (1 + β)^2 (1 + μ)) Sin[φs[t]]^2)^2) -
      (1 + β) Sin[φns[t] - φs[t]] φs'[t]^2))]]
```

```
1/a^4 (-c^2 β (1 + β) Cos[φs[t]] Sin[φns[t]] /
     (m^2 (β^2 Sin[φns[t]]^2 - 2 β (1 + β) Sin[φns[t]] Sin[φs[t]] + (2 + μ + 2 β (1 + μ) + β^2 (1 + μ)) Sin[φs[t]]^2)^2) +
   Sin[φs[t]] (a^3 g (2 + β + μ + β μ) +
     c^2 (2 + μ + 2 β (1 + μ) + β^2 (1 + μ)) Cos[φs[t]] /
      (m^2 (β^2 Sin[φns[t]]^2 - 2 β (1 + β) Sin[φns[t]] Sin[φs[t]] + (2 + μ + 2 β (1 + μ) + β^2 (1 + μ)) Sin[φs[t]]^2)^2)) + (1 +
     β) Cos[φns[t] - φs[t]]
    (-c^2 (1 + β) Cos[φns[t]] Sin[φs[t]] /
      (m^2 (β^2 Sin[φns[t]]^2 - 2 β (1 + β) Sin[φns[t]] Sin[φs[t]] + (2 + μ + 2 β (1 + μ) + β^2 (1 + μ)) Sin[φs[t]]^2)^2) + Sin[φns[t]]
      (-a^3 g + c^2 β Cos[φns[t]] /
        (m^2 (β^2 Sin[φns[t]]^2 - 2 β (1 + β) Sin[φns[t]] Sin[φs[t]] + (2 + μ + 2 β (1 + μ) + β^2 (1 + μ)) Sin[φs[t]]^2)^2)))) -
 β (1 + β) Sin[φns[t] - φs[t]] φns'[t]^2 + 1/2 (1 + β)^2 Sin[2 (φns[t] - φs[t])]
  φs'[t]^2
```

```
FullSimplify[%]
```

```
1/a^4 (-a^3 g (1 + β) Cos[φns[t] - φs[t]] Sin[φns[t]] +
    a^3 g (2 + β + μ + β μ) Sin[φs[t]] + (c^2 (-2 β (1 + β) Sin[φns[t]]^2 Sin[φns[t] - φs[t]] +
         (-(1 + β)^2 Cos[2 φns[t] - φs[t]] + (3 + 2 μ + β (2 + β) (1 + 2 μ)) Cos[φs[t]]) Sin[φs[t]])) /
     (2 m^2 (β^2 Sin[φns[t]]^2 - 2 β (1 + β) Sin[φns[t]] Sin[φs[t]] + (2 + μ + β (2 + β) (1 + μ)) Sin[φs[t]]^2)^2)) -
 β (1 + β) Sin[φns[t] - φs[t]] φns'[t]^2 + 1/2 (1 + β)^2 Sin[2 (φns[t] - φs[t])]
  φs'[t]^2
```

```
Simplify[% / (1 + (1 + β)^2 (1 + μ) - (1 + β)^2 Cos[ϕns[t] - ϕs[t]]^2)]
```

$$\Bigg(\frac{1}{a^4}\Big(-a^3\,g\,(1+\beta)\,\mathrm{Cos}[\phi ns[t]-\phi s[t]]\,\mathrm{Sin}[\phi ns[t]]+ a^3\,g\,(2+\beta+\mu+\beta\,\mu)\,\mathrm{Sin}[\phi s[t]]+\big(c^2\,(-2\,\beta\,(1+\beta)\,\mathrm{Sin}[\phi ns[t]]^2\,\mathrm{Sin}[\phi ns[t]-\phi s[t]]+ (-(1+\beta)^2\,\mathrm{Cos}[2\,\phi ns[t]-\phi s[t]]+(3+2\,\mu+\beta\,(2+\beta)\,(1+2\,\mu))\,\mathrm{Cos}[\phi s[t]])\,\mathrm{Sin}[\phi s[t]])\big)\Big/ \Big(2\,m^2\,(\beta^2\,\mathrm{Sin}[\phi ns[t]]^2-2\,\beta\,(1+\beta)\,\mathrm{Sin}[\phi ns[t]]\,\mathrm{Sin}[\phi s[t]]+(2+\mu+\beta\,(2+\beta)\,(1+\mu))\,\mathrm{Sin}[\phi s[t]]^2)^2\Big)\Big)- \beta\,(1+\beta)\,\mathrm{Sin}[\phi ns[t]-\phi s[t]]\,\phi ns'[t]^2+\frac{1}{2}\,(1+\beta)^2\,\mathrm{Sin}[2\,(\phi ns[t]-\phi s[t])]\,\phi s'[t]^2\Bigg)\Big/ \big(1+(1+\beta)^2\,(1+\mu)-(1+\beta)^2\,\mathrm{Cos}[\phi ns[t]-\phi s[t]]^2\big)$$

Impact Transition Map: General Form to Scalar Form (Normalized)

```
Qn = {{-Beta, -Beta + (meu (1 + Beta) ^2 + 2 (1 + Beta)) Cos[ThetaNS - ThetaS]}, {0, -Beta}};
Qp =  {{Beta (Beta - (1 + Beta) Cos[ThetaNS - ThetaS]), (1 + Beta) ((1 + Beta) - Beta Cos[ThetaNS - ThetaS]) +
     1 + meu (1 + Beta) ^2}, {Beta^2, -Beta (1 + Beta) Cos[ThetaNS - ThetaS]}};
Qpinv =
  Inverse[
   Qp];

H = FullSimplify[Qpinv.Qn];
MatrixForm[H]
```

$$\begin{pmatrix} -\frac{(1+\mathrm{Beta})\,\mathrm{Cos}[\mathrm{ThetaNS}-\mathrm{ThetaS}]}{2+\mathrm{meu}+\mathrm{Beta}\,(2+\mathrm{Beta})\,(1+\mathrm{meu})-(1+\mathrm{Beta})^2\,\mathrm{Cos}[\mathrm{ThetaNS}-\mathrm{ThetaS}]^2} & \frac{-2+(-1+\mathrm{Beta})\,(1+\mathrm{Beta})^2\,\mathrm{meu}+(1+\mathrm{Beta})^2\,(2+\mathrm{meu}+\mathrm{Beta}\,\mathrm{meu})\,\mathrm{Cos}[2\,(\mathrm{ThetaNS}-\mathrm{ThetaS})]}{2\,\mathrm{Beta}\,(2+\mathrm{meu}+\mathrm{Beta}\,(2+\mathrm{Beta})\,(1+\mathrm{meu})-(1+\mathrm{Beta})^2\,\mathrm{Cos}[\mathrm{ThetaNS}-\mathrm{ThetaS}]^2)} \\ -\frac{\mathrm{Beta}}{2+\mathrm{meu}+\mathrm{Beta}\,(2+\mathrm{Beta})\,(1+\mathrm{meu})-(1+\mathrm{Beta})^2\,\mathrm{Cos}[\mathrm{ThetaNS}-\mathrm{ThetaS}]^2} & \frac{2\,(1+\mathrm{Beta})\,(1+\mathrm{meu}+\mathrm{Beta}\,\mathrm{meu})\,\mathrm{Cos}[\mathrm{ThetaNS}-\mathrm{ThetaS}]}{3+2\,\mathrm{meu}+\mathrm{Beta}\,(2+\mathrm{Beta})\,(1+2\,\mathrm{meu})-(1+\mathrm{Beta})^2\,\mathrm{Cos}[2\,(\mathrm{ThetaNS}-\mathrm{ThetaS})]} \end{pmatrix}$$

```
FullSimplify[H.{ThetaPNS, ThetaPS}]
```

$$\{-(2\,\mathrm{Beta}\,(1+\mathrm{Beta})\,\mathrm{ThetaPNS}\,\mathrm{Cos}[\mathrm{ThetaNS}-\mathrm{ThetaS}]- \mathrm{ThetaPS}\,(-2+(-1+\mathrm{Beta})\,(1+\mathrm{Beta})^2\,\mathrm{meu}+(1+\mathrm{Beta})^2\,(2+\mathrm{meu}+\mathrm{Beta}\,\mathrm{meu})\,\mathrm{Cos}[2\,(\mathrm{ThetaNS}-\mathrm{ThetaS})]))\,/ \,(2\,\mathrm{Beta}\,(2+\mathrm{meu}+\mathrm{Beta}\,(2+\mathrm{Beta})\,(1+\mathrm{meu})-(1+\mathrm{Beta})^2\,\mathrm{Cos}[\mathrm{ThetaNS}-\mathrm{ThetaS}]^2)),\; -(\mathrm{Beta}\,\mathrm{ThetaPNS})\,/\,(2+\mathrm{meu}+\mathrm{Beta}\,(2+\mathrm{Beta})\,(1+\mathrm{meu})-(1+\mathrm{Beta})^2\,\mathrm{Cos}[\mathrm{ThetaNS}-\mathrm{ThetaS}]^2)+ (2\,(1+\mathrm{Beta})\,(1+\mathrm{meu}+\mathrm{Beta}\,\mathrm{meu})\,\mathrm{ThetaPS}\,\mathrm{Cos}[\mathrm{ThetaNS}-\mathrm{ThetaS}])\,/ \,(3+2\,\mathrm{meu}+\mathrm{Beta}\,(2+\mathrm{Beta})\,(1+2\,\mathrm{meu})-(1+\mathrm{Beta})^2\,\mathrm{Cos}[2\,(\mathrm{ThetaNS}-\mathrm{ThetaS})])\}$$