Bipedal Walkers: From Three to Two Dimensions via Lagrangian Reduction

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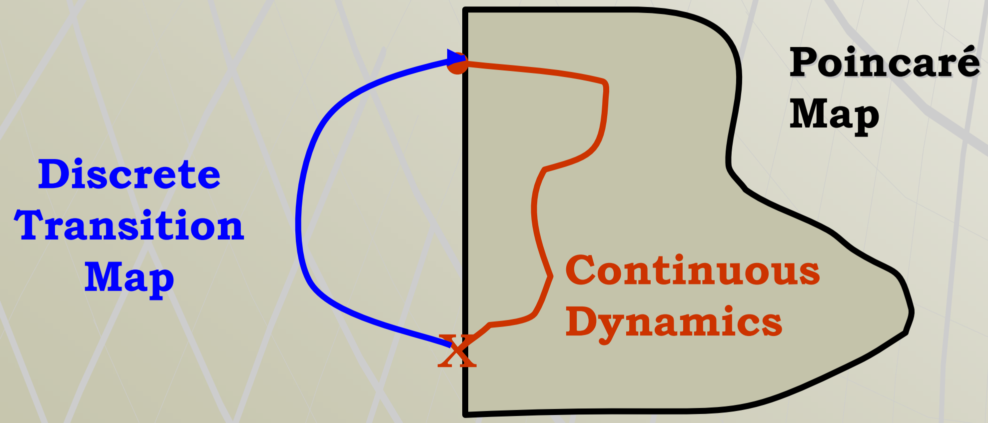
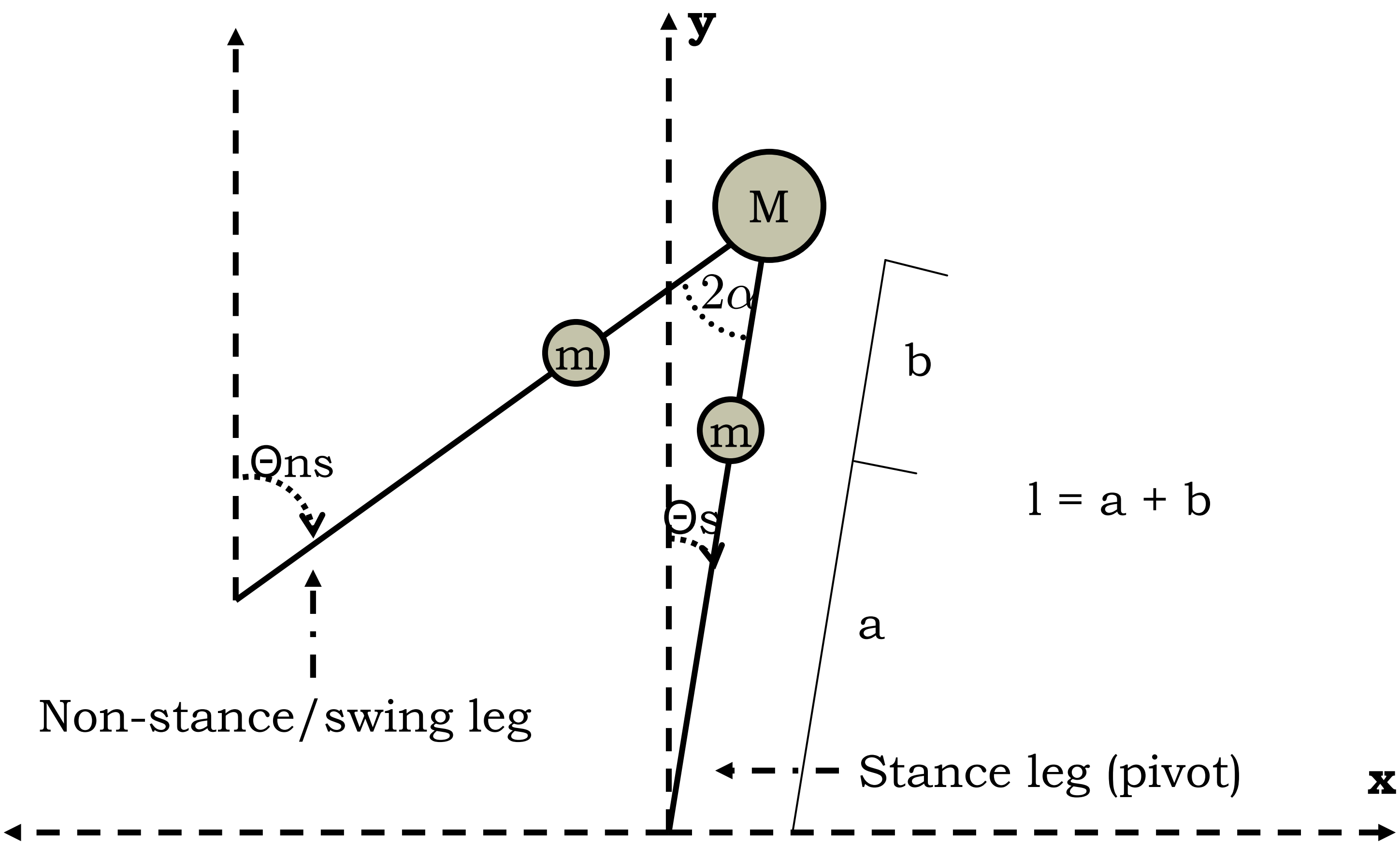
Problem of 3D Walkers

Analysis of 2D Walkers

- Many techniques have already been established for analyzing two dimensional bipedal walkers
- Finding stable walking cycles
  - Dynamics described by non-linear ODEs
  - No straightforward backsolving method to find initial states
  - Solution: Numerical analysis using methods of Poincaré
    - Search feasible phase space for initial states that result in asymptotically stable cycles

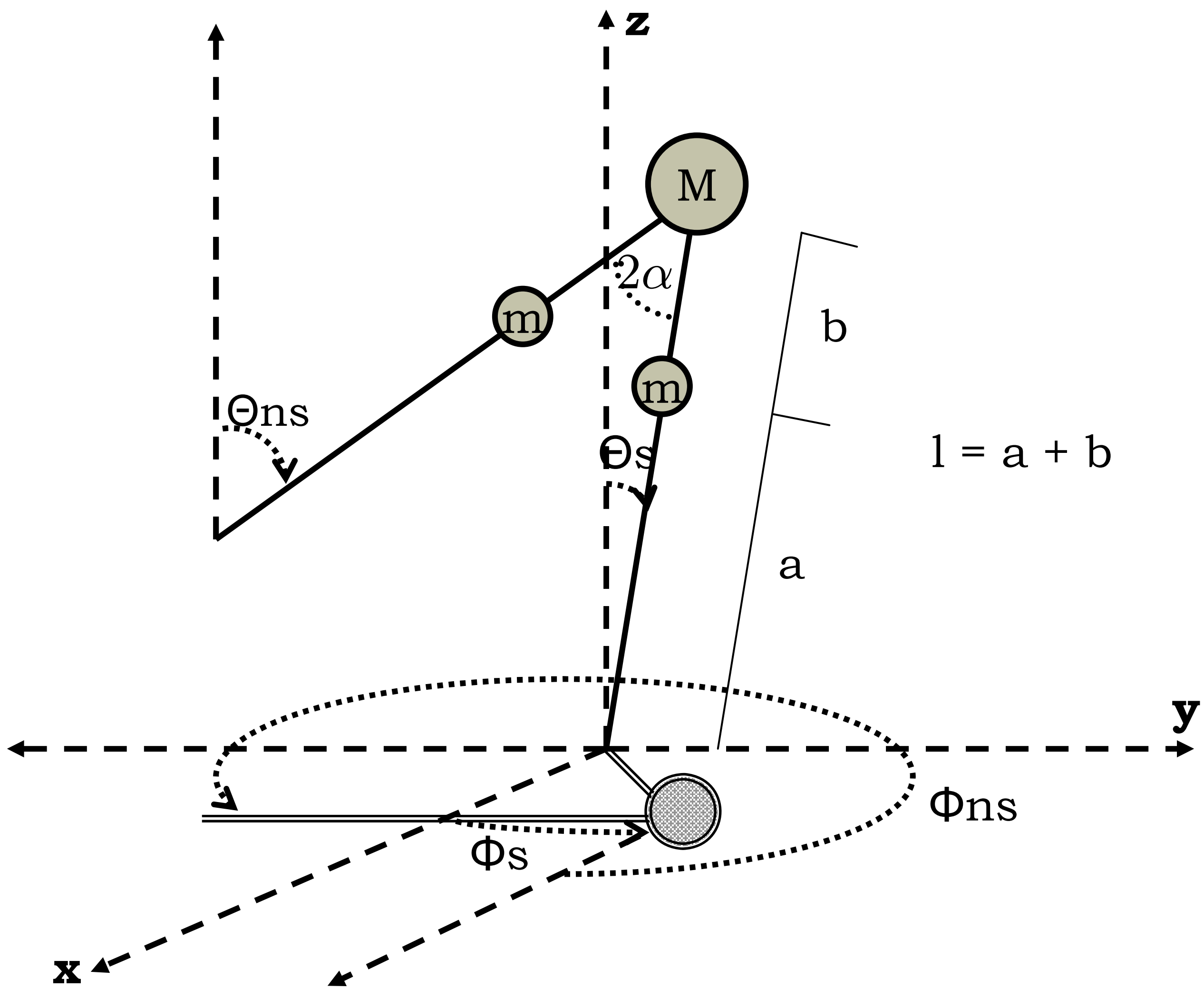


Compass-Gait Bipedal Walker (2D)



❖ Four state dependencies: $\Theta_{non\text{-}stance}$, Θ_{stance}, and time-derivatives

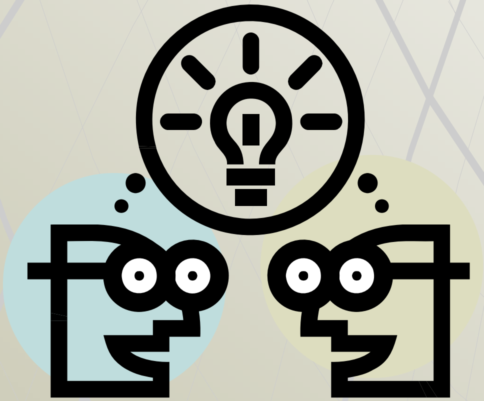
Compass-Gait Bipedal Walker (3D)



❖ Eight state dependencies: $\Theta_{non\text{-}stance}$, Θ_{stance}, $\Phi_{non\text{-}stance}$, Φ_{stance}, and time-derivatives

Scaling Complexity

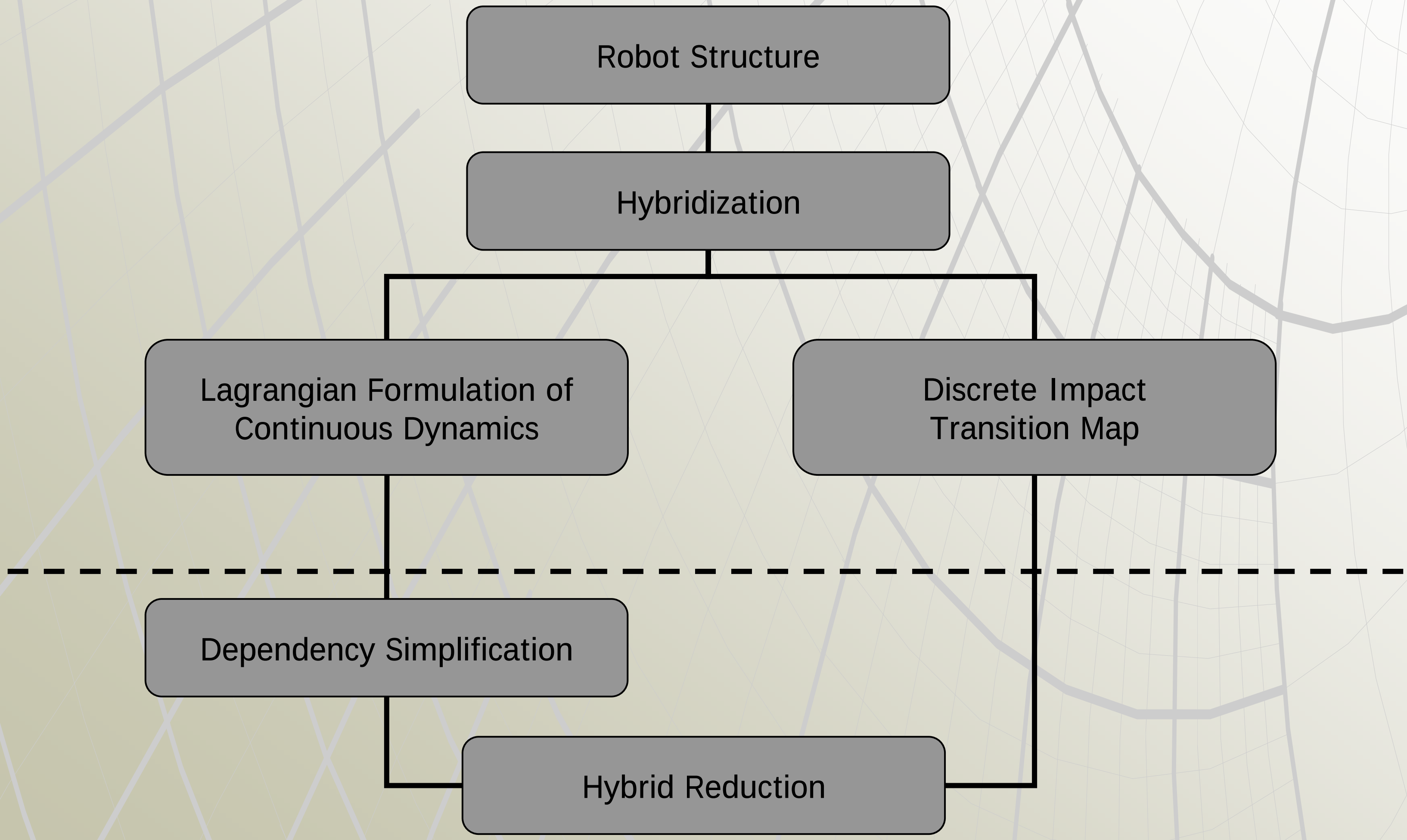
- Increasing the model's dimensions from two to three results in a two-fold increase of state dependency
- Thus, in three dimensions, numerical analysis requires a phase space search of *eight* dimensions
- Analysis is computably impractical!



- **Solution**: Hybrid Reduction on the 3D Model

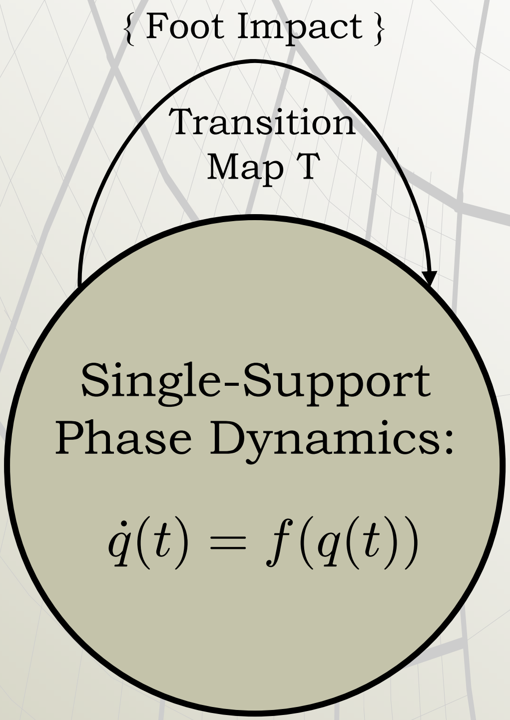
Hybrid Reduction from 3D to 2D

Process of Reduction (General)



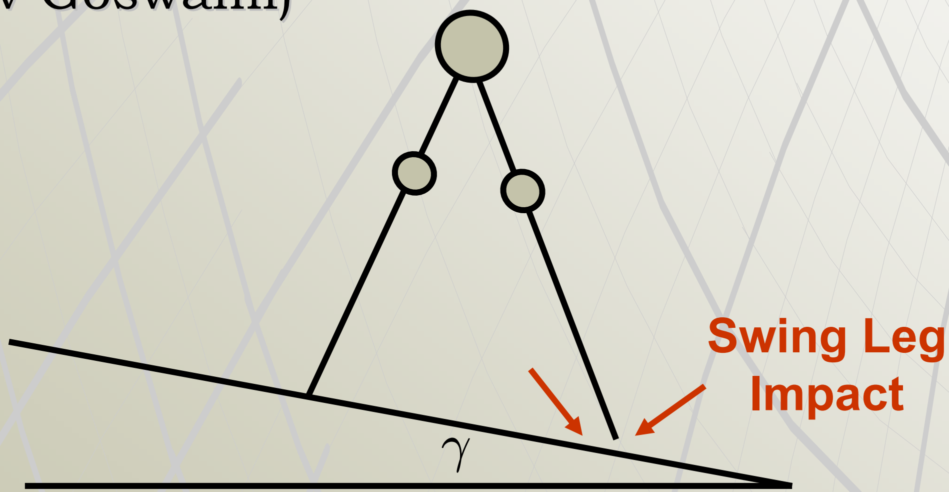
Hybridization

- System's single-support phase guided by differential equations (continuous dynamics)
- Swing leg's impact with ground considered a reset transition for hybrid system (discrete event)



Discrete Foot Impact

- Impact Equations (swing leg impact on ground):
  - Angle positions preserved
  - Discontinuity in angle velocity (different ways of modeling, Grizzle v Goswami)



- Transition Map (hybrid system reset)
  - Swing leg becomes stance leg: angle positions swap
  - Angle velocities: $\dot{\Theta}^{+} = H(\gamma)\, \dot{\Theta}^{-}$

Lagrangian Formulation

- The Lagrangian formulation accounts for all energy in the system
- Lagrangian = Kinetic Energy – Potential Energy

$$L = K - V$$

$$L = \tfrac{1}{2}\, \boldsymbol{\Theta}'^{T} M(\boldsymbol{\Theta})\, \boldsymbol{\Theta}' - \int q(\boldsymbol{\Theta})$$

- Derive the continuous Equations of Motion (passive):

$$M(\boldsymbol{\Theta})\, \boldsymbol{\Theta}'' + F(\boldsymbol{\Theta}, \boldsymbol{\Theta}')\, \boldsymbol{\Theta}' + q(\boldsymbol{\Theta}) = \mathbf{0}$$

where $\boldsymbol{\Theta} = [\Theta ns, \Theta s, \Phi ns, \Phi s]^T$

- M and F are 4x4 matrices and q is a 4x1 vector
- Pages and pages of matrix entries!

Dependency Simplification

- ❖ Goal is to find cyclic variables in Lagrangian

Strategies:

- ❖ Fixing inner angle 2γ => No cyclic variables
- ❖ Limit as M/m approaches infinity => No cyclic
- ❖ Limit as b/a approaches infinity => No cyclic
- ❖ Fixing Φs[t] = Φns[t] (x-y plane)
  - ❑ *Two* cyclic variables: Φns[t] and Φs[t]

M[Θ] =

M1

$$\begin{pmatrix} \beta^2 & -\beta(1+\beta)\,\mathrm{Cos}[\phi ns[t]-\phi s[t]] & 0 & 0 \\ -\beta(1+\beta)\,\mathrm{Cos}[\phi ns[t]-\phi s[t]] & 1+(1+\beta)^2(1+\mu) & 0 & 0 \\ 0 & 0 & \beta^2\,\mathrm{Sin}[\phi ns[t]]^2 & -\beta(1+\beta)\,\mathrm{Sin}[\phi ns[t]]\,\mathrm{Sin}[\phi s[t]] \\ 0 & 0 & -\beta(1+\beta)\,\mathrm{Sin}[\phi ns[t]]\,\mathrm{Sin}[\phi s[t]] & (1+(1+\beta)^2(1+\mu))\,\mathrm{Sin}[\phi s[t]]^2 \end{pmatrix}$$

M2

Routhian Reduction

- $\Phi ns[t]$ and $\Phi s[t]$ independent
- $M_2(\boldsymbol{\Theta})\ \boldsymbol{\Phi}` = \mathbf{c}$ (Routhian constant)

  where $\boldsymbol{\Phi}` = [\ \Phi`ns,\ \Phi`s\ ]^T$,

  $\boldsymbol{\Theta} = [\ \Theta ns,\ \Theta s\ ]^T$,

  $\mathbf{c} = [\ c_1,\ c_2\ ]^T$

  and $\Phi`ns[t] = \Phi`s[t]$
- Solve: $\Phi` = c/m(\boldsymbol{\Theta})$
- Routhian = $[\ L(\boldsymbol{\Theta},\ \boldsymbol{\Theta}`,\ \Phi`) - \boxed{c\ \Phi`}\ ]_{\Phi`=c/m(\boldsymbol{\Theta})}$

  Augmented Term

  $R = \frac{1}{2}\ \boldsymbol{\Theta}`^T\ M_1(\boldsymbol{\Theta})\ \boldsymbol{\Theta}` - \int q(\boldsymbol{\Theta}) - \frac{1}{2}\ c^2/m(\boldsymbol{\Theta})$

Results of Reduction

1.0 Background: Problem of 3D Bipedal Walkers
- 1.1 Analysis of 2D Walkers
- 1.2 Application: Simple Compass-Gait Biped
- 1.3 Scaling Complexity from 2D to 3D

2.0 Hybrid Reduction from 3D to 2D
- 2.1 Hybridization of Robot Motion
- 2.2 Discrete Foot Impact Transition
- 2.3 Lagrangian Continuous Dynamics
- 2.4 Dependency Simplification of Lagrangian
- 2.5 Routhian Reduction

3.0 Results
- **3.1 Reduced Model**
- **3.2 Equations of Motion (2D)**
- **3.3 Hypothesis of 3D Motion**

4.0 Final Thoughts

Reduced Model

- Continous Equations of Motion (passive):

$$M(\boldsymbol{\Theta})\,\boldsymbol{\Theta}'' + F(\boldsymbol{\Theta}, \boldsymbol{\Theta}')\,\boldsymbol{\Theta}' + q(\boldsymbol{\Theta}) + aug_c(\boldsymbol{\Theta}) = \mathbf{0}$$

where $\boldsymbol{\Theta} = [\ \Theta ns, \Theta s\ ]^T$

Original 2D Model

Augmented Potential

Conclusions:

- M and F are 2x2 matrices; q and aug are 2x1 vectors
- The reduced model (now 2D) is equivalent to the original 2D model with an augmented term
- Matrices M and F and vector q remain the same; overall potential term is modified.
- Additional constant c (if zero => original 2D model)
- Uniqueness: Trivial to bring back to unique 3D model

Normalized Eqns of Motion (2D)

$M(\boldsymbol{\Theta}) =$

$$\begin{pmatrix} \beta^2 & -\beta\,(1+\beta)\,\mathrm{Cos}[\phi ns[t]-\phi s[t]] \\ -\beta\,(1+\beta)\,\mathrm{Cos}[\phi ns[t]-\phi s[t]] & 1+(1+\beta)^2\,(1+\mu) \end{pmatrix}$$

$F(\boldsymbol{\Theta}, \boldsymbol{\Theta}`) =$

$$\begin{pmatrix} 0 & -\beta\,(1+\beta)\,\mathrm{Sin}[\phi ns[t]-\phi s[t]]\,\phi s'[t] \\ \beta\,(1+\beta)\,\mathrm{Sin}[\phi ns[t]-\phi s[t]]\,\phi ns'[t] & 0 \end{pmatrix}$$

$$q(\boldsymbol{\Theta}) = \begin{pmatrix} g\,\beta\,\mathrm{Sin}[\phi ns[t]] \\ -g\,(1+(1+\beta)\,(1+\mu))\,\mathrm{Sin}[\phi s[t]] \end{pmatrix}$$

$-aug_c(\boldsymbol{\Theta}) =$

$$\begin{pmatrix} \dfrac{c^2\,\beta\,\mathrm{Cos}[\phi ns[t]]\,(\beta\,\mathrm{Sin}[\phi ns[t]]-(1+\beta)\,\mathrm{Sin}[\phi s[t]])}{(\beta^2\,\mathrm{Sin}[\phi ns[t]]^2-2\,\beta\,(1+\beta)\,\mathrm{Sin}[\phi ns[t]]\,\mathrm{Sin}[\phi s[t]]+(1+(1+\beta)^2\,(1+\mu))\,\mathrm{Sin}[\phi s[t]]^2)^2} \\ \dfrac{c^2\,\mathrm{Cos}[\phi s[t]]\,((-1-\beta)\,\beta\,\mathrm{Sin}[\phi ns[t]]+(1+(1+\beta)^2\,(1+\mu))\,\mathrm{Sin}[\phi s[t]])}{(\beta^2\,\mathrm{Sin}[\phi ns[t]]^2-2\,\beta\,(1+\beta)\,\mathrm{Sin}[\phi ns[t]]\,\mathrm{Sin}[\phi s[t]]+(1+(1+\beta)^2\,(1+\mu))\,\mathrm{Sin}[\phi s[t]]^2)^2} \end{pmatrix}$$

Hypothesis of 3D Motion

- Current reduced model is in 2D, but can easily bring into 3D using the property of Routhian reduction

$$\boldsymbol{\Phi}` = M_2^{-1}(\boldsymbol{\Theta})\,\mathbf{c},$$

$$\boldsymbol{\Phi} = \boldsymbol{\Phi}_0 + \int M_2^{-1}(\boldsymbol{\Theta})\,\mathbf{c}$$

- Hypothesis of Reduced 3D Motion: If stable limit cycles exist for the reduced model in two dimensions, then stable limit cycles also exist for the three dimensional version of the reduced model
- We will be conducting tests with Simon Ng's HyVisual implementation to confirm this hypothesis

Final Thoughts

- A 3D biped model is related to its much simpler 2D model by a computable term
- The 3D model is thus easily implemented in a visual simulation, which is useful for confirming results
- The final outcome of this project is a general framework by which previously established techniques can be applied to three dimensional bipeds