Pricing in Non-cooperative Dynamic Games

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20 August 2012
Competitive Collaboration

• Incentive theory is used to study problems with socio-economic considerations where agents have private information.
• In general, resources are scarce causing competition amongst self interested agents.
• Mechanism design is the tool used to coordinate competitive agents.
• In a dynamic environment, we aim to design prices to shape utilities of competitive agents for the purpose of coordination in engineering problems.
Competitive Collaboration

- Network Security
- Efficient Energy Management in Buildings
- Controlled Diffusion in Multi-Agent Networks

Node Status: Infected → Healthy
Outline

- Problem Formulation
- Convex Program
- Applications
- Future Work
Inducing Cooperation in LQ Games

Selfish Agents

- Agent $i$: state $x_i$, control $u_i$
- Cost:
  \[
  \tilde{J}_i(x_i, u_i) := \int_0^\infty x_i^T Q_i x_i + u_i^T \hat{R}_i u_i \, dt
  \]
- Optimization Problem:
  \[
  \min_{u_i} \tilde{J}_i(x_i, u_i)
  \]
  subj. to: $\dot{x} = Ax + Bu$
- Selfish, rational agents - play Nash
- In general, Nash is suboptimal.

Social Planner

- Societal Cost:
  \[
  J_T(x, u) := \sum_{i=1}^p \tilde{J}_i(x_i, u_i)
  \]
- Optimization Problem:
  \[
  \min_{u} \quad J_T(x, u)
  \]
  subj. to: $\dot{x} = Ax + Bu$
- Solution: standard LQR theory
  \[
  u_i^d = K_i^d x
  \]
- How can we induce selfish agents to use $u_i^d = K_i^d x$?
Quadratic Pricing Mechanism

Social planner adds quadratic price to each agent’s cost

\[ I_i = \int_0^\infty u^T S_i u \, dt \]

New cost for each agent:

\[ J_i = \int_0^\infty x_i^T Q_i x_i + u_i^T \hat{R}_i u_i + u^T S_i u \, dt \]

\[ = \int_0^\infty x_i^T Q_i x_i + u^T R_i u \, dt \]

where \( R_i = \begin{bmatrix} \hat{R}_i & 0 \\ 0 & 0 \end{bmatrix} + S_i \) are the prices.

Problem Statement:

Can leader choose \( \{R_i\}_{i=1}^p \) so that agents are compelled to use \( u_i^d = K_i^d x \)?
Convex Feasibility Problem to Find Prices

Theorem

If there exists $\{R_i\}_{i=1}^P$ and $\{P_i\}_{i=1}^P$ such that the convex problem

\[
\begin{align*}
P_i &> 0 & \text{(PSD $P$ in A.R.E.)} \\
\tilde{R}_i &> 0 \\
\begin{bmatrix}
\tilde{Q}_i & \tilde{N}_i \\
(\tilde{N}_i)^T & \tilde{R}_i
\end{bmatrix} &\succeq 0 & \text{(PD Cost Function)} \\
(\tilde{A}_i)^T P_i + P_i(\tilde{A}_i) + \tilde{Q}_i - (K_i^d)^T \tilde{R}_i K_i^d & = 0 & \text{(Optimality Conditions)} \\
(B_i^T P_i + (\tilde{N}_i)^T) & = -\tilde{R}_i K_i^d
\end{align*}
\]

is feasible, then $\{u_i^* = u_i^d = K_i^d x\}_{i=1}^P$ is a Nash equilibrium of the LQ game, thereby achieving the social planner’s goal.
Revenue Neutral Cost Modifications

Cost at Social Optimum With and Without Pricing

\[ \hat{J}_i^d = \int_0^\infty x_i^T Q_i x_i + (u_i^d)^T \hat{R}_i u_i^d \, dt \quad J_i^d = \int_0^\infty x_i^T Q_i x_i + (u^d)^T R_i u^d \, dt \]

- Prices force desired control, but a priori may have

\[ W \triangleq \sum_{i=1}^{p} (J_i^d - \hat{J}_i^d) \neq 0 \]

- \( W > 0 \implies \) social planner pockets difference
- \( W < 0 \implies \) social planner must cover difference

- Add objective: \((x_0\) is initial state of system)

\[
\min |W| = \begin{cases} 
|\sum_{i=1}^{p} (x_0^T P_i x_0 - \hat{J}_i^d)| & \text{(if } x_0 \text{ is known)} \\
|\sum_{i=1}^{p} (\text{Tr}(P_i) - \hat{J}_i^d)| & \text{(in expectation if } x_0 \text{ is unknown)}
\end{cases}
\]
Applications - Energy Management

- Each zone’s temperature is affected by neighboring zones.
- Each zone has a desired set point temperature.
- Each zone has a local utility function comprised of energy and comfort costs.

Heat transfer model:
\[ \dot{x} = Ax + Bu + d, \text{ where} \]
\[ A_{ii} = - \left( \sum_{j \in N_i} \frac{h_{i,j} a_{i,j}}{\rho V_i C_p} + \frac{\dot{m}_i}{\rho V_i} + \frac{h_{i,o} a_{i,o}}{\rho V_i C_p} \right) \]
\[ A_{ij} = \begin{cases} \frac{h_{i,j} a_{i,j}}{\rho V_i C_p}, & \text{if } j \in N_i \\ 0, & \text{otherwise} \end{cases} \]
and \[ B_{ii} = \frac{\dot{m}_i}{\rho V_i}, \quad d_i = \frac{h_{i,o} a_{i,o}}{\rho V_i C_p} T_\infty. \]

Agent Utility:
\[ \hat{J}_i = \int_0^\infty \left( x_i - \bar{x}_i \right)^T Q_i \left( x_i - \bar{x}_i \right) dt \]
\[ + (u_i - \bar{u}_i)^T \hat{R}_i (u_i - \bar{u}_i) dt \]
Varying Thermal Conductivity Between Zones

Less insulation between zones results in higher savings.

<table>
<thead>
<tr>
<th>$h_i$ = 24, air between zones</th>
</tr>
</thead>
<tbody>
<tr>
<td>Social Cost</td>
</tr>
<tr>
<td>Total</td>
</tr>
<tr>
<td>Savings</td>
</tr>
</tbody>
</table>

$J_i^d$ : player i’s cost with pricing
$\hat{J}_i^d$ : player i’s portion of the social cost
$J_i^{NE}$ : player i’s cost at Nash Equilibrium given their nominal utility.
Varying Agent Preferences

- Vary Q:R ratios – the team and Nash costs vary depending on the ratio of the comfort cost matrices and the nominal energy cost matrices of the agents.

<table>
<thead>
<tr>
<th>Agent</th>
<th>$Q : \hat{R} = 1 : 1$ all agents</th>
<th>$Q : \hat{R} = 10 : 1$ even agents</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Social Cost</td>
<td>Cost w/ Pricing</td>
</tr>
<tr>
<td>2</td>
<td>12.709</td>
<td>12.275</td>
</tr>
<tr>
<td>4</td>
<td>10.703</td>
<td>10.418</td>
</tr>
<tr>
<td>6</td>
<td>12.541</td>
<td>15.403</td>
</tr>
<tr>
<td>8</td>
<td>12.979</td>
<td>12.515</td>
</tr>
<tr>
<td>9</td>
<td>12.430</td>
<td>12.020</td>
</tr>
<tr>
<td>Total</td>
<td>106.994</td>
<td>106.994</td>
</tr>
<tr>
<td>Savings</td>
<td><strong>0.982%</strong></td>
<td></td>
</tr>
</tbody>
</table>
Future Work

• Applying mechanism design theory to real world scenarios such as Sutardja Dai Hall. In this formulation, we consider a VAV type HVAC system and a switched linear model of the thermodynamics (in collaboration with Anil Aswani).

• Relaxing assumptions of the problem such as: perfect state knowledge (both social planner and selfish agents, i.e. allow for observations of the form $y = Cx$), perfect knowledge of player utilities as well as the linear quadratic assumptions.
References


Minimizing Impact of Non-Local Controls

\[ J_i(x_i, u) = \int_0^\infty x_i^T Q_i x_i + \begin{bmatrix} u_i \\ u_{-i} \end{bmatrix}^T \begin{bmatrix} R_{ii}^{ii} & R_{i-i}^{i-i} \\ R_{i-i}^{i-i} & R_{i-i}^{i-i} \end{bmatrix} \begin{bmatrix} u_i \\ u_{-i} \end{bmatrix} \, dt \]

- \( R_{i-i}^{i-i} \): component of pricing mechanism dependent only on other players’ controls.
- \( R_{i-i}^{ii}, R_{i-i}^{i-i} \): components depending on both \( i \) and other players’ controls.

- Add objective

\[
\min \sum_{j \neq i, l \neq i} w_i^j \left\| R_{i}^{jl} \right\|_2 + \sum_{j \neq i} w_i^j \left( \left\| R_{i}^{ij} \right\|_2 + \left\| R_{i}^{ji} \right\|_2 \right) \] (2)

- First term \( \rightarrow \) diagonal entries of \( R_i \), Second term \( \rightarrow \) off diagonal entries.
Constrained Inverse LQR

Original Dynamics and Cost with Pricing for Agent $i$

$$J_i(x_i, u) = \int_0^\infty x_i^T Q_i x_i + u^T R_i u \ dt \quad \dot{x} = Ax + B_i u - u_i + B_i u_i$$

At Nash Equilibrium $u_{-i} = K_{-i} d x$

New Dynamics and Cost for Player $i$ at Nash Equilibrium

$$\dot{x} = \left( A + B_i K_{-i} \right) x + B_i u_i = \tilde{A}_i x + B_i u_i$$

$$J_i(x, u_i) = \int_0^\infty x_i^T Q_i x_i + \begin{bmatrix} u_i \\ K_{-i} x_i \end{bmatrix}^T \begin{bmatrix} R_{ii}^i & R_{i-i}^i \\ R_{i-i}^i & R_{i-i}^{-i} \end{bmatrix} \begin{bmatrix} u_i \\ K_{-i} x_i \end{bmatrix} \ dt$$

$$= \int_0^\infty x^T \left( \begin{bmatrix} Q_i & 0 \\ 0 & 0 \end{bmatrix} + K_{-i}^T R_{i-i}^{-i} K_{-i} \right) x + u_i^T \begin{bmatrix} R_{ii}^i \\ \tilde{R}_i \end{bmatrix} u_i$$

$$+ 2x^T \begin{bmatrix} K_{-i}^T R_{i-i}^i \end{bmatrix} u_i \ dt$$

$$= \int_0^\infty x^T \tilde{Q}_i x + u_i^T \tilde{R}_i u_i + 2x^T \tilde{N}_i u_i \ dt$$