New Directions in Hybrid Systems Theory

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Outline

1. Softwalls
    - J. Adam Cataldo
    - Prof. Edward Lee
2. Computing Reachability Sets
    - Alex Kurzhanskiy
    - Prof. Pravin Varaiya
3. A Categorical Theory of Hybrid Systems
    - Aaron D. Ames
    - Prof. Shankar Sastry

Outline

1 Softwalls

2 Computing Reachability Sets

3 A Categorical Theory of Hybrid Systems

4 Conclusion

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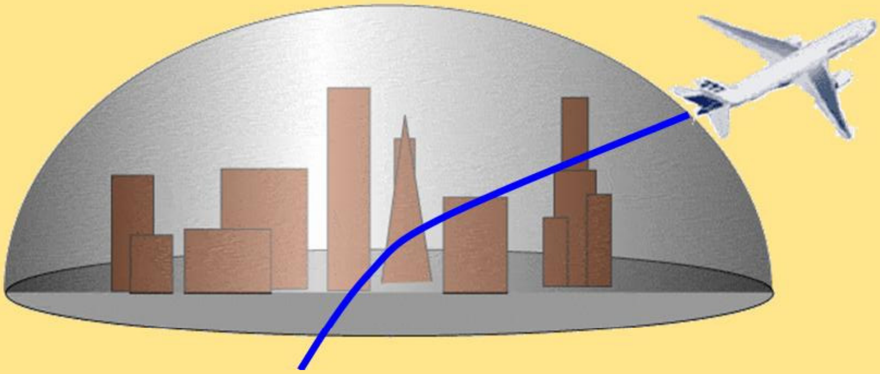
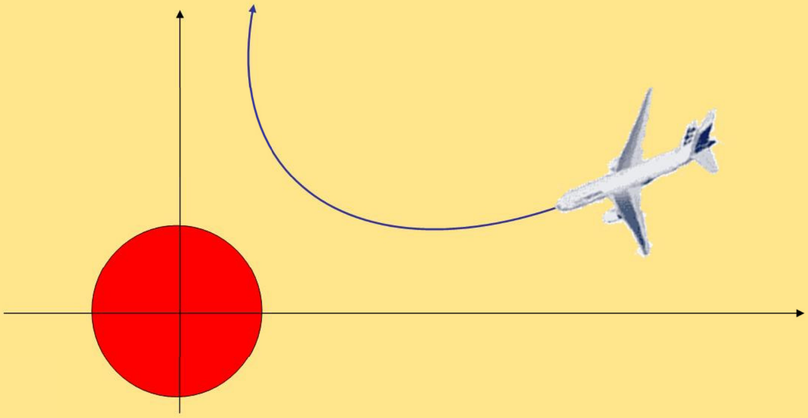
Soft Walls Example: Problem statement

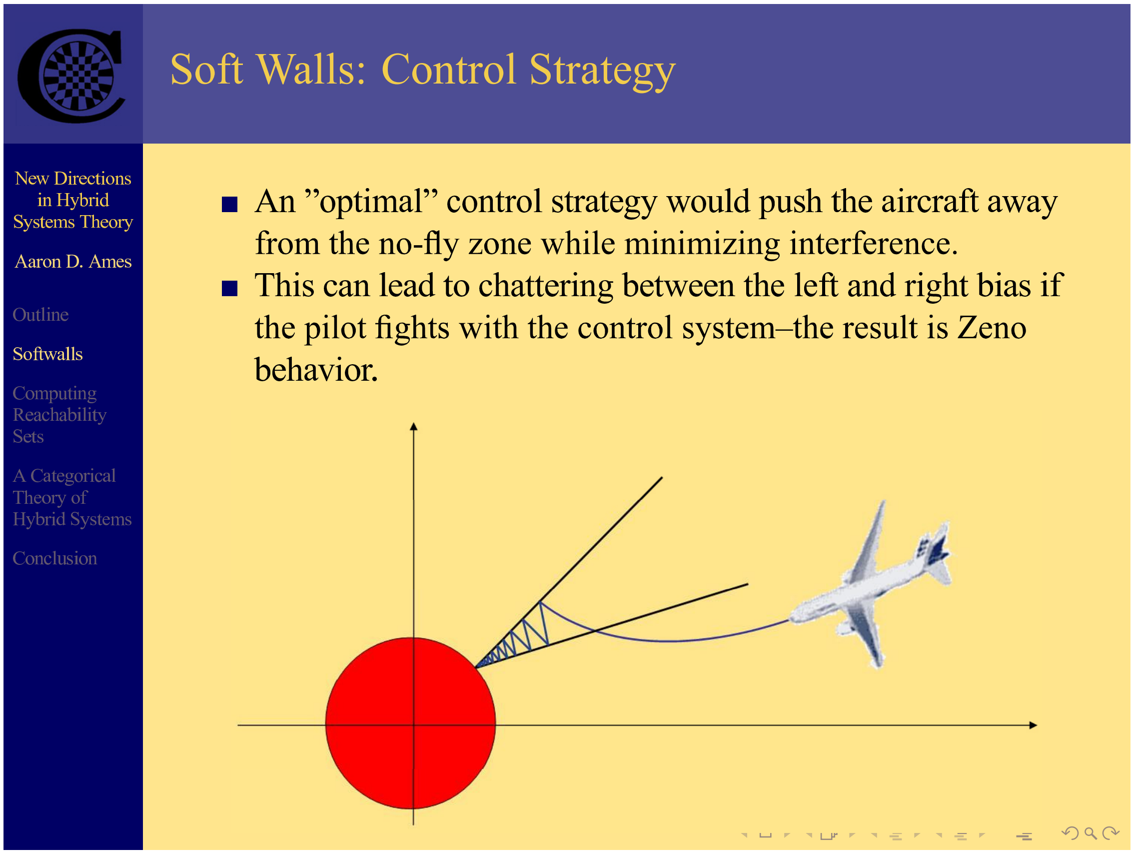
- Enforce no-fly zones using on-board avionics.
- A collision occurs if the aircraft enters a no-fly zone.



Soft Walls: Control Strategy

- An ”optimal” control strategy would push the aircraft away from the no-fly zone while minimizing interference.
- This can lead to chattering between the left and right bias if the pilot fights with the control system–the result is Zeno behavior.





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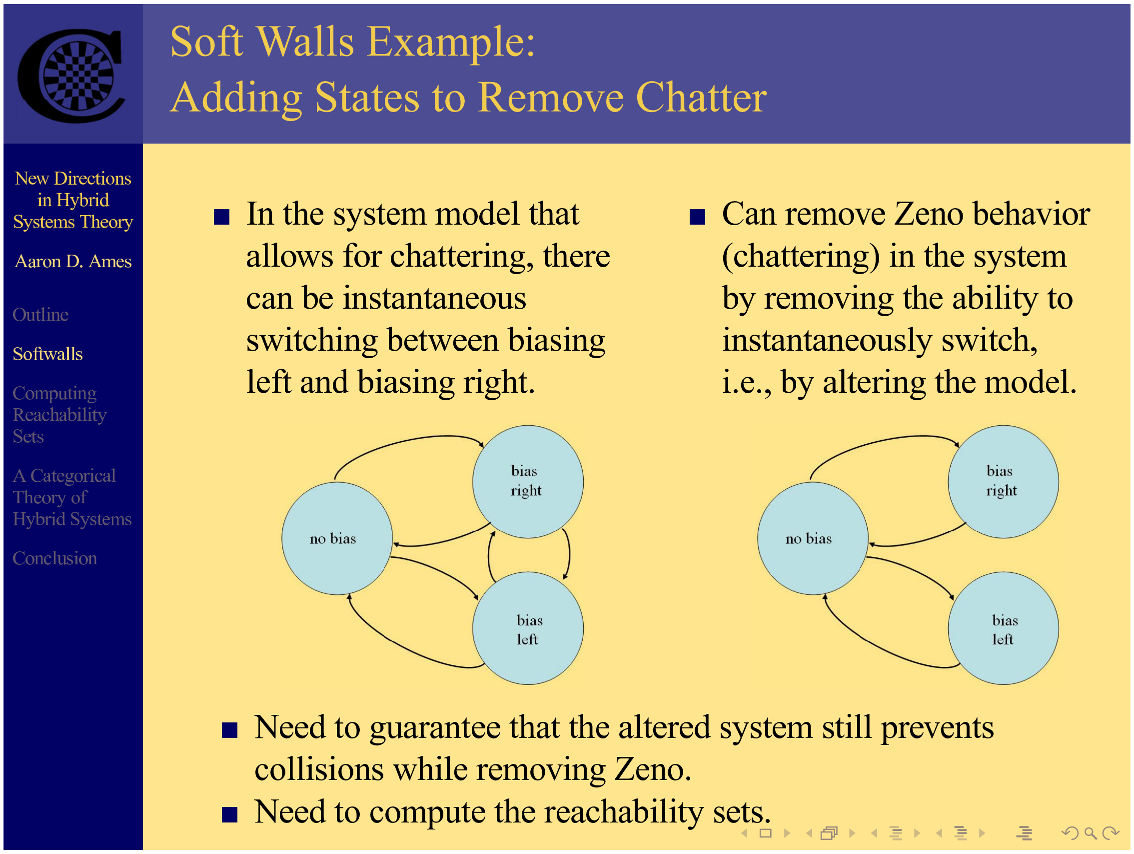

Soft Walls Example: Adding States to Remove Chatter

- In the system model that allows for chattering, there can be instantaneous switching between biasing left and biasing right.
- Can remove Zeno behavior (chattering) in the system by removing the ability to instantaneously switch, i.e., by altering the model.

- Need to guarantee that the altered system still prevents collisions while removing Zeno.
- Need to compute the reachability sets.

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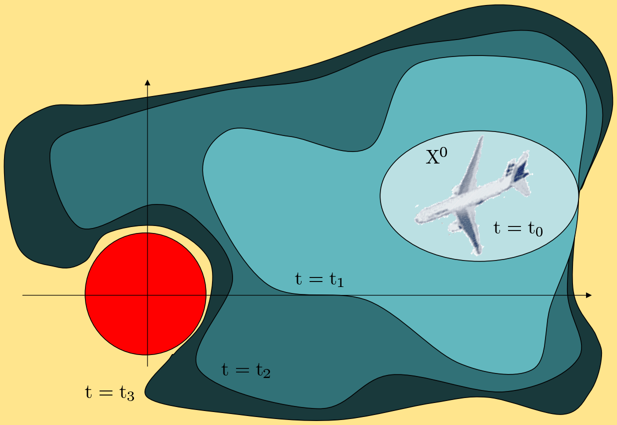
The Reach Set Problem

Problem

Find the reach set $\mathcal{X}(t, t_0, X^0)$ *of all states that can be reached at time* t *starting in* X^0 *at* t_0 *using open loop control* $u(t)$



The Reach Set Problem: Linear Systems

Problem

Find the reach set $\mathcal{X}(t, t_0, X^0)$ of all states that can be reached at time t starting in X^0 at t_0 using open loop control $u(t)$

Consider the linear control system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

where $u(t) \in \mathcal{P}(t)$ and $x(t_0) \in \mathcal{X}^0$ with

$$\mathcal{P}(t) = \mathcal{E}(p(t), P(t)), \quad \mathcal{X}^0 = \mathcal{E}(x(t_0), X^0)$$

where $\mathcal{E}(z, Z) = \{x : (x - z)^T Z(x - z) \leq 1\}$ is an *ellipsoid* with center z and shape matrix Z.

Internal Point Problem

- Calculate the reach set with specification of control for each point of the set.
- Reach the target point at given time t_1.
- Reach the target point in the shortest time.
- Reach the target point at some time in the interval $[t_1, t_2]$.
- If the target point is unreachable, find the nearest reachable point.
- Recursive algorithms (use past computations for future computations to minimize computation costs).

Ellipsoidal Toolbox

- Calculation of reach sets using ellipsoidal approximation algorithms.
- Visualization of their 3D projections.

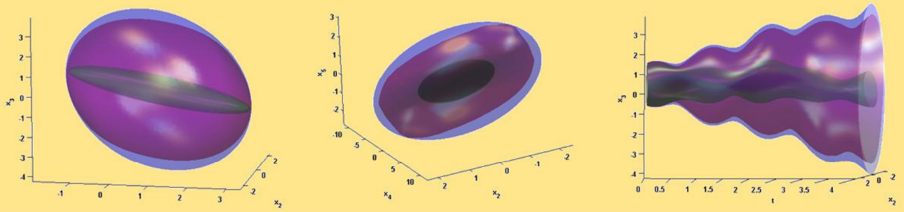


Figure: www.eecs.berkeley.edu/∼akurzhan/ellipsoids

Future Work

- Extend the results to the linear discrete-time systems.
- Extend the functionality of the Ellipsoidal Toolbox:
    - Dynamic projections,
    - Systems with uncertainty.
- Extend the results to hybrid systems.

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A Categorical Theory of Hybrid Systems

- Issues:
  - Current models for hybrid systems are clumsy to manipulate mathematically: they have many different mathematical objects: edges, guards, resets, vector fields.
  - The tools of algebraic topology such as homology theory seem to be well suited to answering questions of existence and uniqueness, non-Zeno behavior of hybrid systems.
  - The ungainliness of the current models does not make this possible.
- Approach: Define hybrid systems using category theory.
  - Define a hybrid system in the formalism of category theory.
  - Preexisting constructions in category theory can be used to apply the tools of algebraic topology and homology theory to hybrid systems.
  - Practical and computable results pertaining to the composition of hybrid systems and Zeno classification can be derived.

Motivation: Morse Theory and Homology

- Homology: Associates to a topological space a sequence of abelian groups.
  - Easier to understand these groups than the topological space.
  - Homology is invariant under deformations of the space.
- Morse theory relates the homology of a space with the behavior of vector fields on that space.
- Question: Can similar results be obtained for hybrid systems?
  - Can a homology theory be developed for hybrid systems?
  - If so, can this homology be used to say useful things about hybrid systems?
- Answer: Yes to all of the above.

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Theorem

If $\mathscr{H}$ is a hybrid system and $\mathbb{H}$ its underlying "space" then

$$\chi(\mathbb{H}) := \sum_{i=0}^{\infty}(-1)^i \dim HH_i(\mathbb{H},\mathbb{R}) = 1 \quad \Rightarrow \quad \mathscr{H} \text{ is not Zeno.}$$

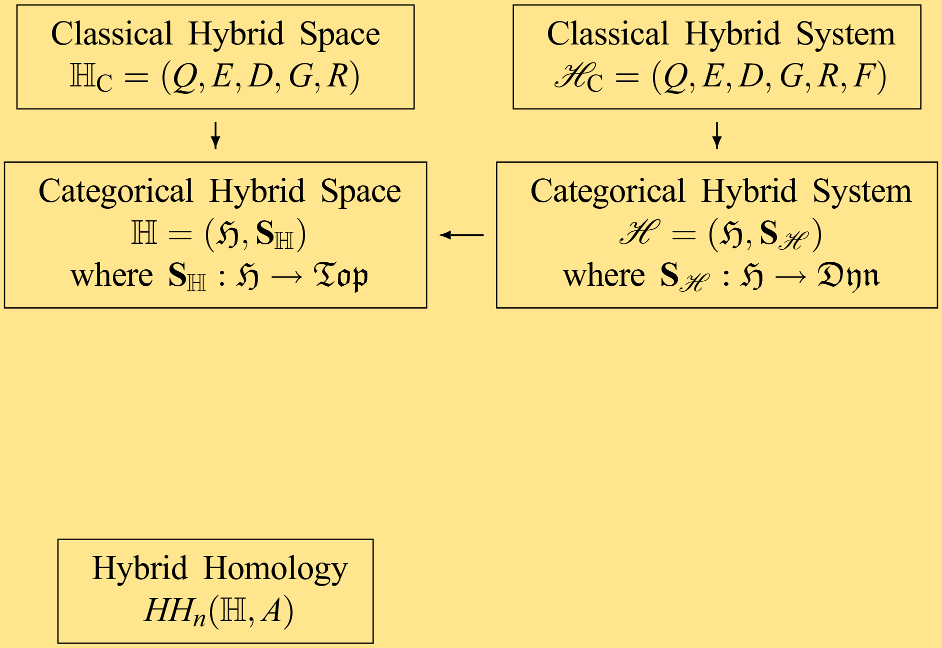
when $\mathbb{H}$ is contractible. Here $HH_i(\mathbb{H},\mathbb{R})$ is the hybrid homology of $\mathbb{H}$.

Outline of Theory:
Getting from Hybrid Systems to Homology

Classical Hybrid System
$\mathscr{H}_{\mathrm{C}} = (Q, E, D, G, R, F)$

Hybrid Homology
$HH_n(\mathbb{H}, A)$

Outline of Theory:
Obtaining Categorical Hybrid Systems



Classical Hybrid Space
$\mathbb{H}_{\mathrm{C}} = (Q, E, D, G, R)$

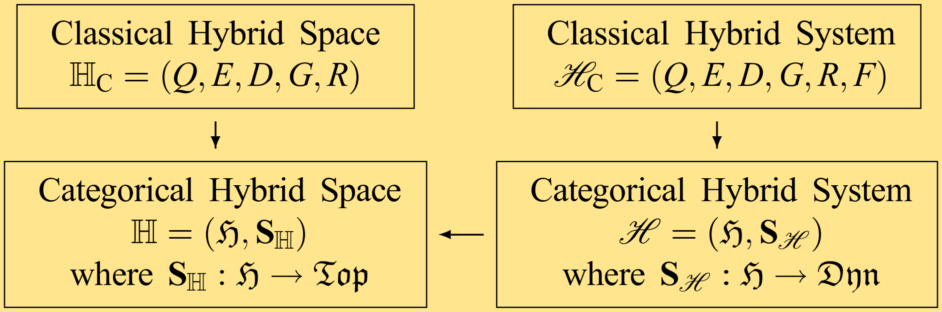
Classical Hybrid System
$\mathscr{H}_{\mathrm{C}} = (Q, E, D, G, R, F)$

Categorical Hybrid Space
$\mathbb{H} = (\mathfrak{H}, \mathbf{S}_{\mathbb{H}})$
where $\mathbf{S}_{\mathbb{H}} : \mathfrak{H} \longrightarrow \mathfrak{Top}$

Categorical Hybrid System
$\mathscr{H} = (\mathfrak{H}, \mathbf{S}_{\mathscr{H}})$
where $\mathbf{S}_{\mathscr{H}} : \mathfrak{H} \longrightarrow \mathfrak{Dyn}$

Hybrid Homology
$HH_n(\mathbb{H}, A)$

Correspondences



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Theorem

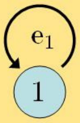
There are injective correspondences

$$\{\text{Classical Hybrid Spaces}\} \longrightarrow \{\text{Categorical Hybrid Spaces}\}$$

$$\{\text{Classical Hybrid Systems}\} \longrightarrow \{\text{Categorical Hybrid Systems}\}$$

Example:
The Classical Bouncing Ball Hybrid Space

- The hybrid system simulating the behavior of a bouncing ball is the standard example of a hybrid system that displays Zeno behavior.
- The hybrid space for the bouncing ball, $\mathbb{H}_B$, is given by:
  - It has as its underlying graph $\Gamma_B = (Q, E)$ given by the diagram:



  - The other elements of the hybrid space are defined as:

$$\begin{aligned} D_{e_1} &= \{(x_1, x_2) : x_1 \geq 0\}, \\ G_{e_1} &= \{(0, x_2) : x_2 \leq 0\}, \\ R_{e_1}(x_1, x_2) &= (0, -cx_2). \end{aligned}$$

Here $0 < c < 1$ is the amount of energy retained in each bounce.

Example:
The Categorical Bouncing Ball Hybrid Space

From the hybrid space for the bouncing ball $\mathbb{H}_B$ we can construct a categorical hybrid space by defining $\mathfrak{H}$ and $\mathbf{S}_{\mathbb{H}_B}$ as follows:

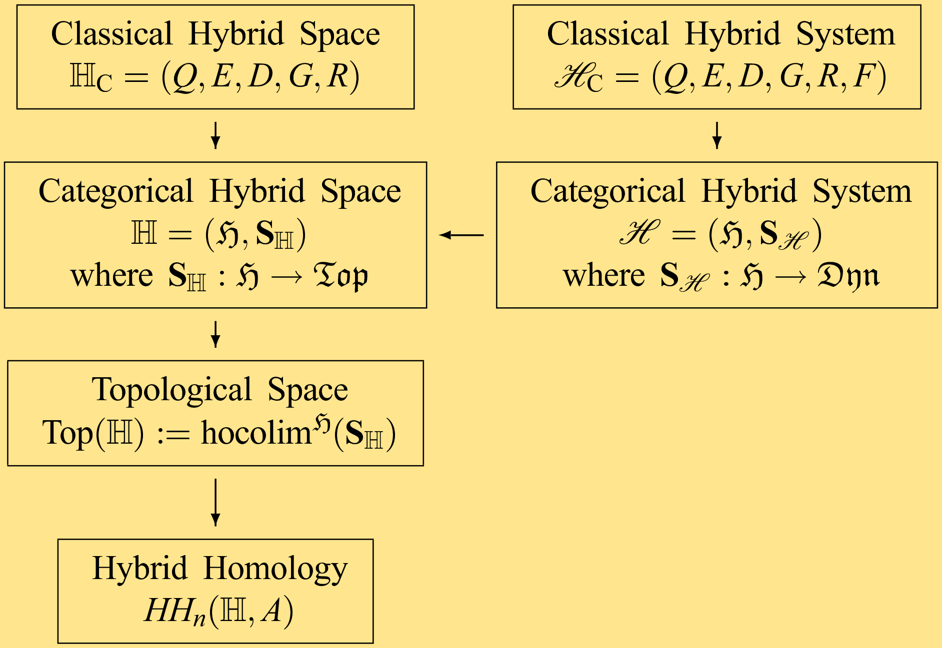
- $\mathfrak{H}$ is given by the diagram:

$$\begin{tikzcd}
e \arrow[d, shift right, "\alpha"'] \arrow[d, shift left, "\beta"] \\
\mathfrak{s}(e) = \mathfrak{t}(e)
\end{tikzcd}$$

- $\mathbf{S}_{\mathbb{H}_B}$ is a functor defined by the diagram of topological spaces:

$$\begin{tikzcd}
e \arrow[dd, shift right, "\alpha"'] \arrow[dd, shift left, "\beta"] & & & \{(0,x_2) : x_2 \leq 0\} \arrow[dd, shift right, "\mathrm{id}"'] \arrow[dd, shift left, "\begin{pmatrix} 0 \\ -cx_2 \end{pmatrix}"] \\
& {} \arrow[r, "\mathbf{S}_{\mathbb{H}_B}"] & {} & \\
\mathfrak{s}(e) = \mathfrak{t}(e) & & & \{(x_1,x_2) : x_1 \geq 0\}
\end{tikzcd}$$

Outline of Theory:
Obtaining a Topological Space



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Categorical Hybrid Space
$\mathbb{H} = (\mathfrak{H}, \mathbf{S}_{\mathbb{H}})$
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Categorical Hybrid System
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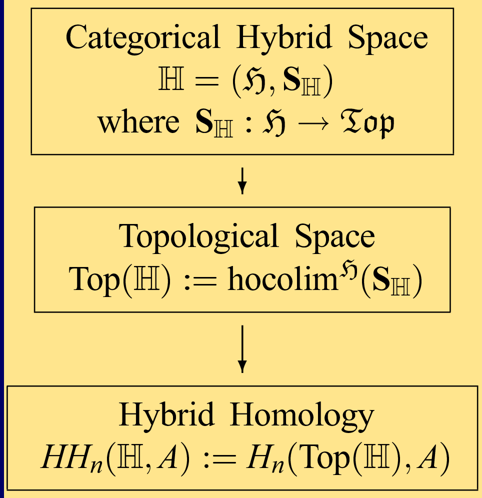
Topological Space
$\mathrm{Top}(\mathbb{H}) := \mathrm{hocolim}^{\mathfrak{H}}(\mathbf{S}_{\mathbb{H}})$

Hybrid Homology
$HH_n(\mathbb{H}, A)$

Topology of Hybrid Systems



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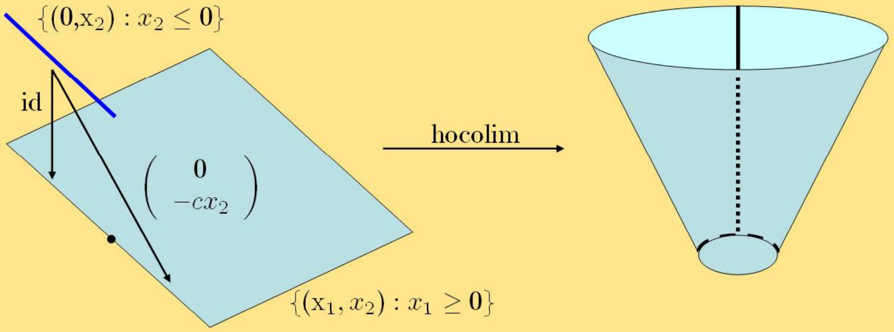
Hybrid Homology
$HH_n(\mathbb{H}, A) := H_n(\mathrm{Top}(\mathbb{H}), A)$

- Associates to a hybrid space a single topological space through well known constructions in algebraic topology.
- Can associate to this topological space its homology; this yields a homology theory for hybrid systems: hybrid homology.
- This topological space and its homology give useful information about the hybrid system.

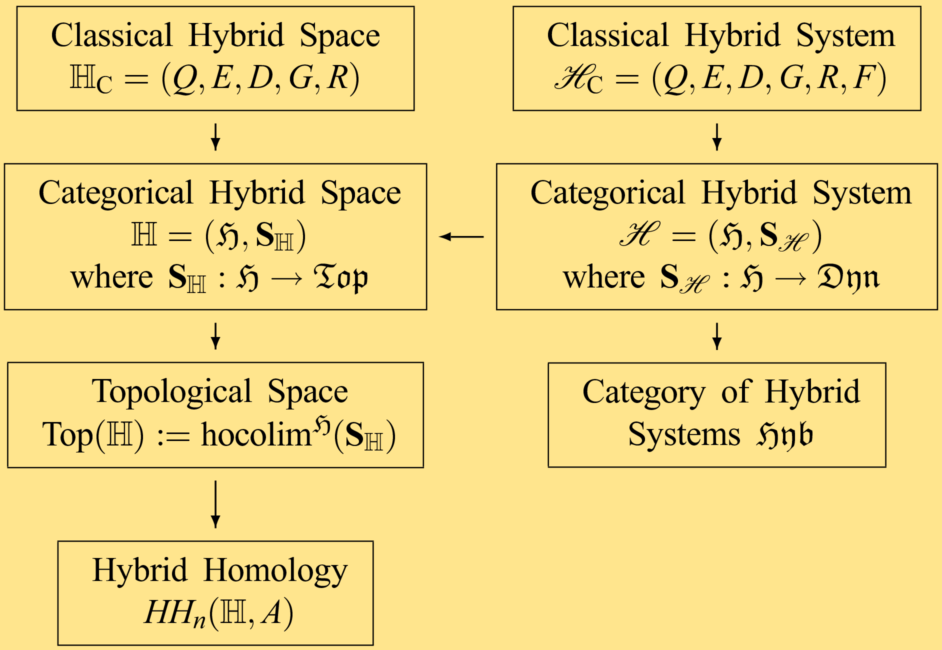
Example:
The Topological Space of the Bouncing Ball

The underlying topological space of the bouncing ball is given by:

$$\mathrm{Top}(\mathbb{H}_B) = \mathrm{hocolim}^{\mathfrak{H}}(\mathbf{S}_{\mathbb{H}_B}) = \text{Punctured Cone}.$$



Outline of Theory: Obtaining the Category of Hybrid Systems



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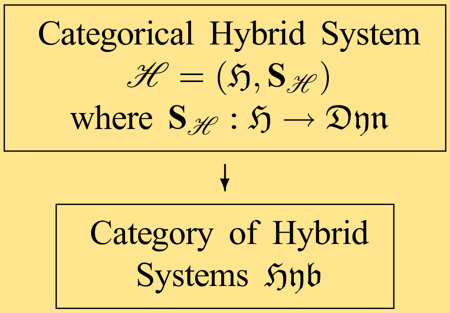
Category of Hybrid Systems $\mathfrak{Hyb}$

Hybrid Homology
$HH_n(\mathbb{H}, A)$

The Category of Hybrid Systems

- The category of hybrid systems is a ”graph” whose vertices are hybrid systems and whose edges are morphisms between hybrid systems.
- Gives the proper framework for understanding multiple hybrid systems and their interactions.
- Allows the mature subject of category theory to be applied to hybrid systems.



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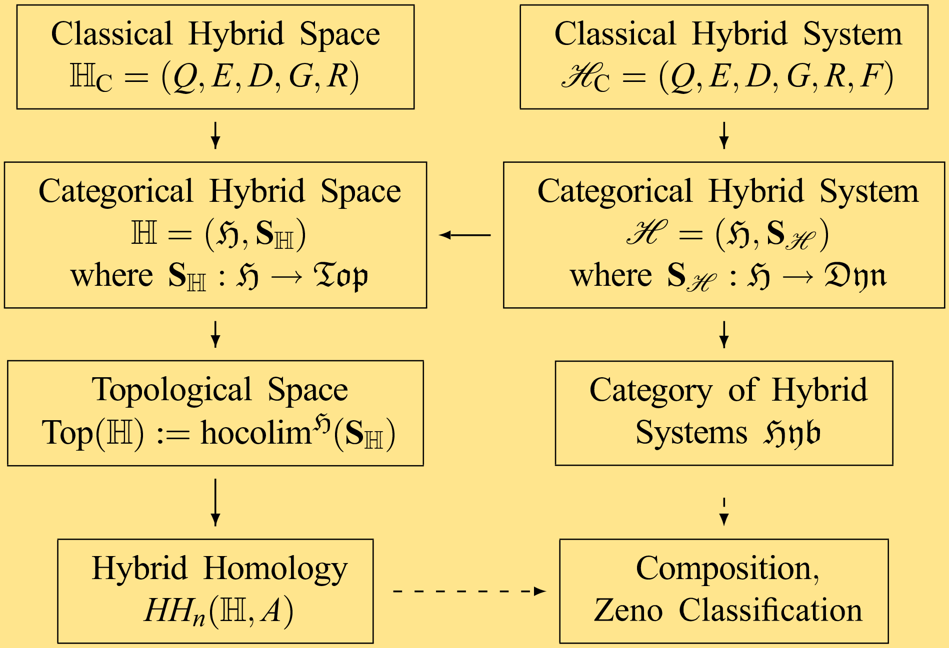
Category of Hybrid Systems $\mathfrak{Hyb}$

Outline of Theory: Deliverables



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Category of Hybrid Systems $\mathfrak{Hyb}$

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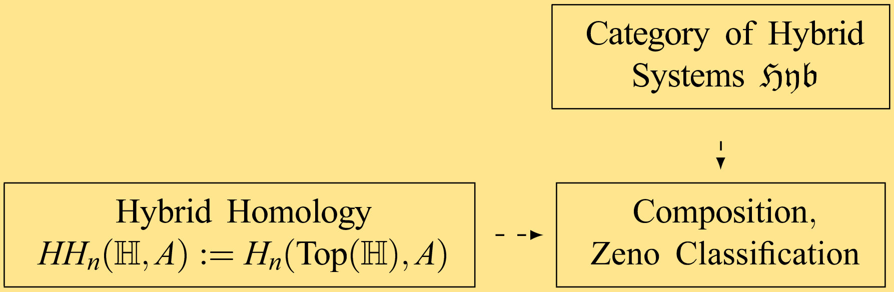
Composition, Zeno Classification

Deliverables

- The category of hybrid systems gives the proper mathematical framework for composing hybrid systems:
  - Composition as the product or coproduct in this category.
- Classification of Zeno through algebraic and homological methods.
- Goal: algebraic necessary and sufficient conditions on whether a given hybrid system is Zeno.



Category of Hybrid Systems $\mathfrak{Hyb}$

Hybrid Homology
$HH_n(\mathbb{H}, A) := H_n(\mathrm{Top}(\mathbb{H}), A)$

Composition, Zeno Classification

Zeno Classification through Homology

Theorem

Let Γ be the underlying graph of a hybrid system $\mathscr{H}$ and $\mathbb{H}$ its underlying hybrid space. Then,

$$\dim(HH_1(\mathbb{H},\mathbb{R})) = \dim(\mathcal{N}(K_\Gamma)) = 0 \quad \Rightarrow \quad \mathscr{H} \text{ is not Zeno.}$$

Here $\mathcal{N}(K_\Gamma)$ is the nullspace of the incidence matrix, K_Γ, of the graph Γ.

Note that $\dim HH_1(\mathbb{H}_B,\mathbb{R}) = 1$, so we cannot conclude that the bouncing ball is not Zeno (which is good since it is Zeno).

$$\begin{tikzcd} \boxed{\begin{array}{c}\text{Hybrid Homology}\\ HH_n(\mathbb{H},A) := H_n(\mathrm{Top}(\mathbb{H}),A)\end{array}} \arrow[r, dashed] & \boxed{\text{Zeno Classification}} \end{tikzcd}$$

Zeno Classification through Homology

Corollary

If $\mathscr{H}$ is a hybrid system and $\mathbb{H} = (\mathfrak{H}, \mathbf{S}_{\mathbb{H}})$ its underlying hybrid space then

$$\chi(\mathbb{H}) = |\mathfrak{Ob}(\mathfrak{H})| - |\mathfrak{Mor}(\mathfrak{H})| = 1 \quad \Rightarrow \quad \mathscr{H} \text{ is not Zeno.}$$

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Conclusion

- The research directions outlined here range from the practical to the theoretical:
  - Practical: Real world Applications (softwalls) and toolboxes for hybrid systems,
  - Theoretical: New directions in hybrid systems theory.
- The practical will affect the theoretical and vice versa:

Eric Temple Bell

“Abstractness, sometimes hurled as a reproach at mathematics, is its chief glory and its surest title to practical usefulness.”