Structure and Interpretation of Signals and Systems
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Chapter 12

Stability

The four Fourier transforms prove to be useful tools for analyzing signals and systems. When a system is LTI, it is characterized by its frequency response H, and its input x and output y are related simply by

$$\forall \omega \in Reals, \quad Y(\omega) = H(\omega)X(\omega),$$

where Y is the Fourier transform of y, and X is the Fourier transform of x.

However, we ignored a lurking problem. Any of the three Fourier transforms, X, Y, or H, may not exist. Suppose for example that x is a discrete-time signal. Then its Fourier transform (the DTFT) is given by

$$\forall \; \omega \in Reals, \quad X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-i\omega n}. \tag{12.1}$$

This is an infinite sum, properly viewed as the limit

$$\forall \; \omega \in Reals, \quad X(\omega) = \lim_{N\to\infty} \sum_{n=-N}^{N} x(n)e^{-i\omega n}. \tag{12.2}$$

As with all such limits, there is a risk that it does not exist. If the limit does not exist for any $\omega \in Reals$, then the Fourier transform becomes mathematically treacherous at best (involving, for example, Dirac delta functions), and mathematical nonsense at worst.

Example 12.1: Consider the sequence

$$x(n) = \begin{cases} 0, & n \le 0 \\ a^{n-1}, & n > 0 \end{cases},$$

where $a > 1$ is a constant. Plugging into (12.1), the Fourier transform should be

$$\forall \; \omega \in Reals, \quad X(\omega) = \sum_{n=0}^{\infty} a^{n-1}e^{-i\omega n}.$$

At $\omega = 0$, it is easy to see that this sum is infinite (every term in the sum is greater than or equal to one). At other values of ω, there are also problems. For example, at $\omega = \pi$,

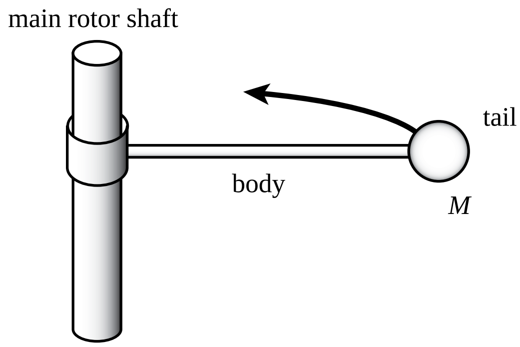

Figure 12.1: A highly simplified helicopter.

the terms of the sum alternate in sign and increase in magnitude as n gets larger. The limit (12.2) clearly will not exist.

A similar problem arises with continuous-time signals. If x is a continuous-time signal, then its Fourier transform (the CTFT) is given by

$$\forall\, \omega \in Reals, \quad X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt. \tag{12.3}$$

Again, there is a risk that this integral does not exist.

This chapter studies signals for which the Fourier transform does not exist. Such signals prove to be both common and useful. The signal in example 12.1 gives the bank balance of example 5.12 when an initial deposit of one dollar is made, and no further deposits or withdrawals are made (thus, it is the impulse response of the bank account). This signal grows without bound, and any signal that grows without bound will cause difficulties when using the Fourier transform.

The bank account is said to be an **unstable system**, because its output can grow without bound even when the input is always bounded. Such unstable systems are common, so it is unfortunate that the frequency domain methods we have studied so far do not appear to apply.

Example 12.2: A helicopter is intrinsically an unstable system, requiring an electronic or mechanical feedback control system to stabilize it. It has two rotors, one above, which provides lift, and one on the tail. Without the rotor on the tail, the body of the helicopter would start to spin. The rotor on the tail counteracts that spin. However, the force produced by the tail rotor must perfectly counter the friction with the main rotor, or the body will spin.

A highly simplified version of the helicopter is shown in figure 12.1. The body of the helicopter is modeled as a horizontal arm with moment of intertia M. The tail rotor goes on the end of this arm. The body of the helicopter rotates freely around the main rotor shaft. Friction with the main rotor will tend to cause it to rotate by applying a torque as suggested by the curved arrow. The tail rotor will have to counter that torque to keep the body of the helicopter from spinning.

Let the input x to the system be the net torque on the tail of the helicopter, as a function of time. That is, at time t, $x(t)$ is the difference between the frictional torque exerted by the main rotor shaft and the counteracting torque exerted by the tail rotor. Let the output y be the velocity of rotation of the body. From basic physics, torque equals moment of inertial times rotational acceleration. The rotational acceleration is $\dot{y}$, the derivative of y, so

$$\dot{y}(t) = x(t)/M.$$

Integrating both sides, assuming that the initial velocity of rotation is zero, we get the output as a function of the input,

$$\forall\, t \in Reals, \quad y(t) = \frac{1}{M}\int_0^t x(\tau)d\tau.$$

It is now easy to see that this system is unstable. Let the input be $x = u$, where u is the **unit step**, given by

$$\forall\, t \in Reals, \quad u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}. \tag{12.4}$$

This input is clearly bounded. It never exceeds one in magnitude. However, the output grows without bound.

In practice, a helicopter uses a feedback system to determine how much torque to apply at the tail rotor to keep the body of the helicopter straight. We will see how to do this in chapter 14.

In this chapter we develop the basics of modeling unstable systems in the frequency domain. We define two new transforms, called the **Z transform** and **Laplace transform**. The Z transform is a generalization of the DTFT and applies to discrete-time signals. The Laplace transform is a generalization of the CTFT and applies to continuous-time signals. These generalizations support frequency-domain analysis of signals that do not have a Fourier transform, and thus allow analysis of unstable systems.

In particular, let $\hat{X}$ denote the Laplace or Z transform of x, depending on whether it is a continuous or discrete-time signal. Then the Laplace or Z transform of the output of an LTI system is given by $\hat{Y} = \hat{H}\hat{X}$, where $\hat{H}$ is the Laplace or Z transform of the impulse response. This relation applies even when the system is unstable. Thus, these transforms take the place of the Fourier transform when the Fourier transform cannot be used. $\hat{H}$ is called the **transfer function** of the LTI system, and it is a generalization of the frequency response.

12.1 Boundedness and stability

In this section, we identify a simple condition for the existence of the DTFT, which is that the signal be **absolutely summable**. We then define a **stable system** and show that an LTI system is stable if and only if its impulse response is absolutely summable. Continuous-time signals are slightly more complicated, requiring slightly more than that they be **absolutely integrable**. The conditions for the existence of the CTFT are called the **Dirichlet conditions**, and once again, if the impulse response of an LTI system satisfies these conditions, then it is stable.

12.1.1 Absolutely summable and absolutely integrable

A discrete-time signal x is said to be **absolutely summable** if

$$\sum_{n=-\infty}^{\infty} |x(n)|$$

exists and is finite. The "absolutely" in "absolutely summable" refers to the absolute value (or magnitude) in the summation. The sum is said to **converge absolutely**. The following simple fact gives a condition for the existence of the DTFT:

> If a discrete-time signal x is absolutely summable, then its DTFT exists and is finite for all ω.

To see that this is true, note that the DTFT exists and is finite if and only if

$$\forall\, \omega \in Reals, \quad |X(\omega)| = \left| \sum_{n=-\infty}^{\infty} x(n) e^{-i\omega n} \right|$$

exists and is finite. But

$$\left| \sum_{n=-\infty}^{\infty} x(n) e^{-i\omega n} \right| \leq \sum_{n=-\infty}^{\infty} |x(n) e^{-i\omega n}| \tag{12.5}$$

$$= \sum_{n=-\infty}^{\infty} |x(n)| \cdot |e^{-i\omega n}| \tag{12.6}$$

$$= \sum_{n=-\infty}^{\infty} |x(n)|. \tag{12.7}$$

This follows from the following facts about complex (or real) numbers:

$$|a+b| \leq |a| + |b|,$$

which is known as the **triangle inequality** (and generalizes to infinite sums),

$$|ab| = |a| \cdot |b|,$$

and

$$\forall\, \theta \in Reals, \quad |e^{i\theta}| = 1.$$

We can conclude from (12.5) that

$$\forall\, \omega \in Reals, \quad |X(\omega)| \leq \sum_{n=-\infty}^{\infty} |x(n)|.$$

This means that if x is absolutely summable, then the DTFT exists and is finite. It follows from the fact that if a sum converges absolutely, then it also converges (without the absolute value).

A continuous-time signal x is said to be **absolutely integrable** if

$$\int_{-\infty}^{\infty} |x(t)| dt$$

exists and is finite. A similar argument to that above (with summations replaced by integrals) suggests that if a continuous-time signal x is absolutely integrable, then its CTFT should exist and be finite for all ω. However, caution is in order. Integrals are more complicated than summations, and we need some additional conditions to ensure that the integral is well defined. We can use essentially the same conditions given on page 234 for the convergence of the continuous-time Fourier series. These are called the **Dirichlet conditions**, and require three things:

- x is absolutely integrable;
- in any finite interval, x is of **bounded variation**, meaning that there are no more than a finite number of maxima or minima; and
- in any finite interval, x is continuous at all but a finite number of points.

Most any signal of practical engineering importance satisfies the last two conditions, so the important condition is that it be absolutely integrable. We will henceforth assume without comment that all continuous-time signals satisfy the last two conditions, so the only important condition becomes the first one. Under this assumption, the following simple fact gives a condition for the existence of the CTFT:

> An absolutely integrable continuous-time signal x has a CTFT X, and its CTFT $X(\omega)$ is finite for all $\omega \in Reals$.

12.1.2 Stability

A system is said to be **bounded-input bounded-output stable** (**BIBO stable** or just **stable**) if the output signal is bounded for all input signals that are bounded.

Consider a discrete-time system with input x and output y. An input is bounded if there is a real number $M < \infty$ such that $|x(k)| \leq M$ for all $k \in Integers$. An output is bounded if there is a real number $N < \infty$ such that $|y(n)| \leq N$ for all $n \in Integers$. The system is stable if for any input bounded by M, there is some bound N on the output.

Consider a discrete-time LTI system with impulse response h. The output y corresponding to the input x is given by the convolution sum,

$$\forall n \in Integers, \quad y(n) = (h * x)(n) = \sum_{m=-\infty}^{\infty} h(m)x(n-m). \tag{12.8}$$

Suppose that the input x is bounded with bound M. Then, applying the triangle inequality, we see that

$$|y(n)| \le \sum_{m=-\infty}^{\infty} |h(m)||x(n-m)| \le M \sum_{m=-\infty}^{\infty} |h(m)|.$$

Thus, if the impulse response is absolutely summable, then the output is bounded with bound

$$N = M \sum_{m=-\infty}^{\infty} |h(m)|.$$

Thus, if the impulse response of an LTI system is absolutely summable, then the system is stable. The converse is also true, but more difficult to show. That is, if the system is stable, then the impulse response is absolutely summable (see box on page 399). The same argument applies for continuous-time signals, so in summary:

> A discrete-time LTI system is stable if and only if its impulse response is absolutely summable. A continuous-time LTI system is stable if and only if its impulse response is absolutely integrable.

The following example makes use of the **geometric series identity**, valid for any real or complex a where $|a| < 1$,

$$\sum_{m=0}^{\infty} a^m = \frac{1}{1-a}. \tag{12.9}$$

To verify this identity, just multiply both sides by $1-a$ to get

$$\sum_{m=0}^{\infty} a^m - a \sum_{m=0}^{\infty} a^m = 1.$$

This can be written

$$a^0 + \sum_{m=1}^{\infty} a^m - \sum_{m=1}^{\infty} a^m = 1.$$

Now note that $a^0 = 1$ and that the two sums are identical. Since $|a| < 1$, the sums converge, and hence they cancel, so the identity is true.

Example 12.3: As in example 12.1, the impulse response of the bank account of example 5.12 is

$$h(n) = \begin{cases} 0, & n \le 0 \\ a^{n-1}, & n > 0 \end{cases},$$

Probing further: Stable systems and their impulse response

Consider a discrete-time LTI system with real-valued impulse response h. In this box, we show that if the system is stable, then its impulse response is absolutely summable. To show this, we show the contrapositive.[a] That is, we show that if the impulse response is not absolutely summable, then the system is not stable. To do this, suppose that the impulse response is not absolutely summable. That is, the sum

$$\sum_{n=-\infty}^{\infty} |h(n)|$$

is not bounded. To show that the system is not stable, we need only to find one bounded input for which the output either does not exist or is not bounded. Such an input is given by

$$\forall\, n \in Integers, \quad x(n) = \begin{cases} h(-n)/|h(-n)|, & h(n) \neq 0 \\ 0, & h(n) = 0 \end{cases}$$

This input is clearly bounded, with bound $M = 1$. Plugging this input into the convolution sum (12.8) and evaluating at $n = 0$ we get

$$\begin{aligned} y(0) &= \sum_{m=-\infty}^{\infty} h(m)x(-m) \\ &= \sum_{m=-\infty}^{\infty} (h(m))^2/|h(m)| \\ &= \sum_{m=-\infty}^{\infty} |h(m)|, \end{aligned}$$

where the last step follows from the fact that for real-valued $h(m)$, $(h(m))^2 = |h(m)|^2$. But since the impulse response is not absolutely summable, $y(0)$ does not exist or is not finite, so the system is not stable.

A nearly identical argument works for continuous-time systems.

[a] The contrapositive of a statement "if p then q" is "if not q then not p." The contrapositive is true if and only if the original statement is true.

where $a > 1$ is a constant that reflects the interest rate. This impulse response is not absolutely summable, so this system is not stable. A system with the same impulse response, but where $0 < a < 1$, however, would be stable, as is easily verified using (12.9). To use this identity, note that

$$\begin{aligned}\sum_{n=-\infty}^{\infty} |h(n)| &= \sum_{n=1}^{\infty} a^{n-1} \\ &= \sum_{m=0}^{\infty} a^{m} \\ &= \frac{1}{1-a},\end{aligned}$$

where the second step results from a change of variables, letting $m = n-1$.

Example 12.4: Consider a continuous-time LTI system with impulse response

$$\forall\, t \in Reals, \quad h(t) = a^t u(t),$$

where $a > 0$ is a real number and u is the unit step, given by (12.4). To determine whether this system is stable, we need to determine whether the impulse response is absolutely integrable. That is, we need to determine whether the following integral exists and is finite,

$$\int_{-\infty}^{\infty} |a^t u(t)| dt.$$

Since $a > 0$ and u is the unit step, this simplifies to

$$\int_{0}^{\infty} a^t dt.$$

From calculus, we know that this integral is infinite if $a \geq 1$, so the system is unstable if $a \geq 1$. The integral is finite if $0 < a < 1$ and is equal to

$$\int_{0}^{\infty} a^t dt = -1/\ln(a).$$

Thus, the system is stable if $0 < a < 1$.

As we see, when all pertinent signals are absolutely summable (or absolutely integrable), then we can use Fourier transform techniques with confidence. However, many useful signals do not fall in this category (the unit step and sinusoidal signals, for example). Moreover, many useful systems have impulse responses that are not absolutely summable (or absolutely integrable). Fortunately, we can generalize the DTFT and CTFT to get the Z transform and Laplace transform, which easily handle signals that are not absolutely summable.

12.2 The Z transform

Consider a discrete-time signal x that is not absolutely summable. Consider the scaled signal x_r given by

$$\forall\, n \in Integers, \quad x_r(n) = x(n)r^{-n}, \tag{12.10}$$

for some real number $r \geq 0$. Often, this signal is absolutely summable when r is chosen appropriately. This new signal, therefore, will have a DTFT, even if x does not.

Example 12.5: Continuing with example 12.3, the impulse response of the bank account is

$$h(n) = \begin{cases} 0, & n \leq 0 \\ a^{n-1}, & n > 0 \end{cases},$$

where $a > 1$. This system is not stable. However, the scaled signal

$$h_r(n) = h(n)r^{-n}$$

is absolutely summable if $r > a$. Its DTFT is

$$\begin{aligned}
\forall r > a, \forall \omega \in Reals, \quad H_r(\omega) &= \sum_{m=-\infty}^{\infty} h(m) r^{-m} e^{-i\omega m} \\
&= \sum_{m=1}^{\infty} a^{m-1}(re^{i\omega})^{-m} \\
&= \sum_{n=0}^{\infty} a^{n}(re^{i\omega})^{-n-1} \\
&= (re^{i\omega})^{-1} \sum_{n=0}^{\infty} (a(re^{i\omega})^{-1})^{n} \\
&= \frac{(re^{i\omega})^{-1}}{1 - a(re^{i\omega})^{-1}}.
\end{aligned}$$

The second step is by change of variables, $n = m - 1$, and the final step applies the geometric series identity (12.9).

In general, the DTFT of the scaled signal x_r in (12.10) is

$$\forall\, \omega \in Reals, \quad X_r(\omega) = \sum_{m=-\infty}^{\infty} x(m)(re^{i\omega})^{-m}.$$

Notice that this is a function not just of ω, but also of r, and in fact, we are only sure it is valid for values of r that yield an absolutely summable signal h_r. If we define the complex number

$$z = re^{i\omega}$$

then we can write this DTFT as

$$\boxed{\forall\, z \in RoC(x), \quad \hat{X}(z) = \sum_{m=-\infty}^{\infty} x(m) z^{-m},} \tag{12.11}$$

where $\hat{X}$ is a function called the **Z transform** of x,

$$\hat{X}\colon RoC(x) \to Complex$$

where $RoC(x) \subset Complex$ is given by

$$\boxed{RoC(x) = \{z = re^{i\omega} \in Complex \mid x(n)r^{-n} \text{ is absolutely summable.}\}} \tag{12.12}$$

The term RoC is shorthand for **region of convergence**.

Example 12.6: Continuing example 12.5, we can recognize from the form of $H_r(\omega)$ that the Z transform of the impulse response h is

$$\forall\, z \in RoC(h), \quad \hat{H}(z) = \frac{z^{-1}}{1 - az^{-1}} = \frac{1}{z-a},$$

where the last step is the result of multiplying top and bottom by z. The RoC is

$$RoC(h) = \{z = re^{i\omega} \in Complex \mid r > a\}$$

The Z tranform $\hat{H}$ of the impulse response h of an LTI system is called the **transfer function** of the system.

12.2.1 Structure of the region of convergence

When a signal has a Fourier transform, then knowing the Fourier transform is equivalent to knowing the signal. The signal can be obtained from its Fourier transform, and the Fourier transform can be obtained from the signal. The same is true of a Z transform, but there is a complication. The Z transform is a function $\hat{X}\colon RoC \to Complex$, and it is necessary to know the set RoC to know the function $\hat{X}$. The region of convergence is a critical part of the Z transform. We will see that very different signals can have very similar Z transforms that differ only in the region of convergence.

Given a discrete-time signal x, $RoC(x)$ is defined to be the set of all complex numbers $z = re^{i\omega}$ for which the following series converges:

$$\sum_{m=-\infty}^{\infty} |x(m)r^{-m}|.$$

Notice that if this series converges, then so does

$$\sum_{m=-\infty}^{\infty} |x(m)z^{-m}|$$

for any complex number z with magnitude r. This is because

$$|x(m)z^{-m}| = |x(m)(re^{i\omega})^{-m}| = |x(m)| \cdot |r^{-m}| \cdot |e^{-i\omega m}| = |x(m)| \cdot |r^{-m}|.$$

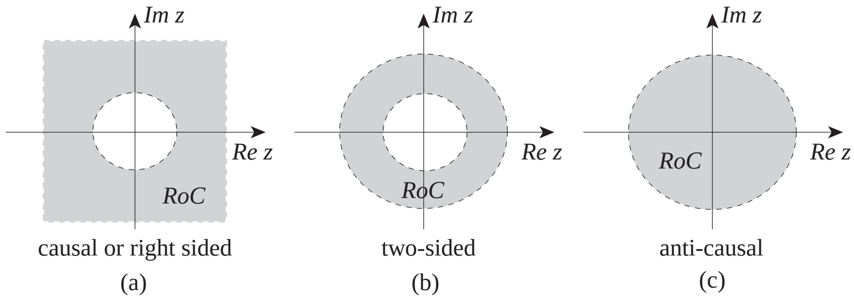

Figure 12.2: Three possible structures for the region of convergence of a Z transform.

Thus, the set *RoC* could equally well be defined to be the set of all complex numbers z such that $x(n)z^{-n}$ is absolutely summable.

Notice that whether this series converges depends only on r, the magnitude of the complex number $z = re^{i\omega}$, and not on ω, its angle. Thus, if any point $z = re^{j\omega}$ is in the set *RoC*, then all points z' with the same magnitude are also in *RoC*. This implies that the set *RoC*, a subset of *Complex*, will have circular symmetry.

The set *RoC* turns out to have even more structure. There are only three possible patterns, illustrated by the shaded areas in figure 12.2. Each figure illustrates the complex plane, and the shaded area is a region of convergence. Each possibility has circular symmetry, in that whether a point is in the *RoC* depends only on its magnitude.

Figure 12.2(a) shows the *RoC* of a causal signal. A discrete-time signal x is **causal** if $x(n) = 0$ for all $n < 0$. The *RoC* is the set of complex numbers $z = re^{j\omega}$ where following series converges:

$$\sum_{m=-\infty}^{\infty} |x(m)r^{-m}|.$$

But if x is causal, then

$$\sum_{m=-\infty}^{\infty} |x(m)r^{-m}| = \sum_{m=0}^{\infty} |x(m)r^{-m}|.$$

If this series converges for some given r, then it must also converge for any $\tilde{r} > r$ (because for all $m \geq 0$, $\tilde{r}^{-m} < r^{-m}$. Thus, if $z \in RoC$, then the *RoC* must include all points in the complex plane on the circle passing through z and every point outside that circle.

Note further that not only must the *RoC* include every point outside the circle, but the series must also converge in the limit as z goes to infinity. Thus, for example, $H(z) = z$ cannot be the Z transform of a causal signal because its *RoC* cannot possibly include infinity ($H(z)$ is not finite there).

Figure 12.2(c) shows the *RoC* of an anti-causal signal. A discrete-time signal x is **anti-causal** if $x(n) = 0$ for all $n > 0$. By a similar argument, if $z \in RoC$, then the *RoC* must include all points in the complex plane on the circle passing through z and every point inside that circle.

Figure 12.2(b) shows the *RoC* of a signal that is neither causal nor anti-causal. Such a signal is called a **two-sided signal**. Such a signal can always be expressed as a sum of a causal signal and an anti-causal signal. The *RoC* is the intersection of the regions of convergence for these two components. To see this, just note that the *RoC* is the set of complex numbers $z = re^{j\omega}$ where following series converges:

$$\sum_{m=-\infty}^{\infty} |x(m)r^{-m}| = \sum_{m=-\infty}^{-1} |x(m)r^{-m}| + \sum_{m=0}^{\infty} |x(m)r^{-m}|.$$

The first sum on the right corresponds to an anti-causal signal, and the second sum on the right to a causal signal. For this series to converge, both sums must converge. Thus, for a two-sided signal, the *RoC* has a ring structure.

Example 12.7: Consider the discrete-time **unit step** signal u, given by

$$u(n) = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases} . \tag{12.13}$$

The Z transform is, using geometric series identity (12.9),

$$\hat{U}(z) = \sum_{m=-\infty}^{\infty} u(m)z^{-m} = \sum_{m=0}^{\infty} z^{-m} = \frac{1}{1-z^{-1}} = \frac{z}{z-1} ,$$

with domain

$$RoC(u) = \{z \in Complex \mid \sum_{m=1}^{\infty} |z|^{-m} < \infty\} = \{z \mid |z| > 1\}.$$

This region of convergence has the structure of figure 12.2(a), where the dashed circle has radius one (that circle is called the **unit circle**). Indeed, this signal is causal, so this structure makes sense.

Example 12.8: The signal v given by

$$v(n) = \begin{cases} -1, & n < 0 \\ 0, & n \geq 0 \end{cases} ,$$

has Z transform

$$\hat{V}(z) = \sum_{m=-\infty}^{\infty} v(m)z^{-m} = -\sum_{m=-\infty}^{1} z^{-m} = -z\sum_{k=0}^{\infty} z^{k} = \frac{z}{z-1} ,$$

with domain

$$RoC(v) = \{z \in Complex \mid \sum_{m=-\infty}^{1} |z|^{-m} < \infty\} = \{z \mid |z| < 1\}.$$

This region of convergence has the structure of figure 12.2(c), where the dashed circle is again the unit circle. Indeed, this signal is anti-causal, so this structure makes sense.

Notice that although the Z transform $\hat{U}$ of u and $\hat{V}$ of v have the *same* algebraic form, namely, $z/(z-1)$, they are *different* functions, because their domains are different. Thus the Z transform of a signal comprises *both* the algebraic form of the Z transform as well as its *RoC*.

A **right-sided signal** x is where for some integer N,

$$x(n) = 0, \quad \forall\, n < N.$$

Of course, if $N \geq 0$, then this signal is also causal. However, if $N < 0$, then the signal is two sided. Suppose $N < 0$. Then we can write the Z transform of x as

$$\sum_{m=-\infty}^{\infty} |x(m)r^{-m}| = \sum_{m=N}^{-1} |x(m)r^{-m}| + \sum_{m=0}^{\infty} |x(m)r^{-m}|.$$

The left summation on the right side is finite, and each term is finite for all $z \in Complex$, so therefore it converges for all $z \in Complex$. Thus, the region of convergence is determined entirely by the right summation, which is the Z transform of the causal part of x. Thus, the region of convergence of a right-sided signal has the same form as that of a causal sequence, as shown in figure 12.2(a). (However, if the signal is not causal, the Z transform does not converge at infinity.)

A **left-sided signal** x is where for some integer N,

$$x(n) = 0, \quad \forall\, n > N.$$

Of course, if $N \leq 0$, then this signal is also anti-causal. However, if $N > 0$, then the signal is two sided. Suppose $N > 0$. Then we can write the Z transform of x as

$$\sum_{m=-\infty}^{\infty} |x(m)r^{-m}| = \sum_{m=-\infty}^{0} |x(m)r^{-m}| + \sum_{m=1}^{N} |x(m)r^{-m}|.$$

The right summation is finite, and therefore converges for all $z \in Complex$ except $z = 0$, where the individual terms of the sum are not finite. Thus, the region of convergence is that of the left summation, except for the point $z = 0$. Thus, the region of convergence of a left-sided signal has the same form as that of an anti-causal sequence, as shown in figure 12.2(c), except that the origin ($z = 0$) is excluded. This, of course, is simply the structure of 12.2(b) where the inner circle has zero radius.

Some signals have no meaningful Z transform.

Example 12.9: The signal x with $x(n) = 1$, for all n, does not have a Z transform. We can write $x = u - v$, where u and v are defined in the previous examples. Thus, the region of convergence of x must be the intersection of the regions of convergence of u and v. However, these two regions of convergence have an empty intersection, so $RoC(x) = \emptyset$.

Viewed another way, the set $RoC(x)$ is the set of complex numbers z where

$$\sum_{m=-\infty}^{\infty} |x(m)z^{-m}| = \sum_{m=-\infty}^{\infty} |z^{-m}| < \infty.$$

But there is no such complex number z.

Note that the signal x in example 12.9 is periodic with any integer period p (because $x(n+p) = x(n)$ for any $p \in Integers$). Thus, it has a Fourier series representation. In fact, as shown in section10.6.3, a periodic signal also has a Fourier transform representation, as long as we are willing to allow Dirac delta functions in the Fourier transform. (Recall that this means that there are values of ω where $X(\omega)$ will not be finite.) With periodic signals, the Fourier series is by far the simplest frequency-domain tool to use. The Fourier transform can also be used if we allow Dirac delta functions. The Z transform, however, is more problematic, because the region of convergence is empty.

12.2.2 Stability and the Z transform

If a discrete-time signal x is absolutely summable, then it has a DTFT X that is finite for all $\omega \in Reals$. Moreover, the DTFT is equal to the Z transform evaluated on the unit circle,

$$\boxed{\forall\, \omega \in Reals, \quad X(\omega) = \hat{X}(z)|_{z=e^{i\omega}} = \hat{X}(e^{i\omega}).}$$

The complex number $z = e^{i\omega}$ has magnitude one, and therefore lies on the unit circle. Recall that an LTI system is stable if and only if its impulse response is absolutely summable. Thus,

> A discrete-time LTI system with impulse response h is stable if and only if the transfer function $\hat{H}$, which is the Z transform of h, has a region of convergence that includes the unit circle.

Example 12.10: Continuing example 12.6, the transfer function of the bank account system has region of convergence given by

$$RoC(h) = \{z = re^{i\omega} \in Complex \mid r > a\},$$

where $a > 1$. Thus, the region of convergence includes only complex numbers with magnitude greater than one, and therefore does not include the unit circle. The bank account system is therefore not stable.

12.2.3 Rational Z tranforms and poles and zeros

All of the Z transforms we have seen so far are **rational polynomials** in z. A rational polynomial is simply the ratio of two finite-order polynomials. For example, the bank account system has transfer function

$$\hat{H}(z) = \frac{1}{z-a}$$

(see example 12.6). The unit step of example 12.7 and its anti-causal cousin of example 12.8 have Z transforms given by

$$\hat{U}(z) = \frac{z}{z-1}, \quad \hat{V}(z) = \frac{z}{z-1},$$

albeit with different regions of convergence.

In practice, most Z transforms of practical interest can be written as the ratio of two finite order polynomials in z,

$$\hat{X}(z) = \frac{A(z)}{B(z)}.$$

The **order** of the polynomial A or B is the power of the highest power of z. For the unit step, the numerator polynomial is $A(z) = z$, a first-order polynomial, and the denominator is $B(z) = z - 1$, also a first-order polynomial.

Recall from algebra that a polynomial of order N has N (possibly complex-valued) **roots**, which are values of z where the polynomial evaluates to zero. The roots of the numerator A are called the **zeroes** of the Z transform, and the roots of the denominator B are called the **poles** of the Z transform. The term "zero" refers to the fact that the Z transform evaluates to zero at a zero. The term "pole" suggests an infinitely high tent pole, where the Z transform evaluates to infinity. The locations in the complex plane of the poles and zeros turn out to yield considerable insight about a Z transform. A plot of these locations is called a **pole-zero plot**. The poles are shown as crosses and the zeros as circles.

Example 12.11: The unit step of example 12.7 and its anti-causal cousin of example 12.8 have pole-zero plots shown in figure 12.3. In each case, the Z transform has the form

$$\frac{z}{z-1} = \frac{A(z)}{B(z)},$$

where $A(z) = z$ and $B(z) = z - 1$. $A(z)$ has only one root, at $z = 0$, so the Z transforms each have one zero, at the origin in the complex plane. $B(z)$ also has only one root, at $z = 1$, so the Z transform has one pole, at $z = 1$.

These plots also show the unit circle, with a dashed line, and the regions of convergence for the two examples, as shaded areas. Note that $RoC(u)$ has the form of a region of convergence of a causal signal, as it should, and $RoC(v)$ has the form of a region of convergence of an anti-causal signal, as it should (see figure 12.2). Note that neither RoC includes the unit circle, so if these signals were impulse responses of LTI systems, then these systems would be unstable.

Consider a rational Z transform

$$\hat{X}(z) = \frac{A(z)}{B(z)}.$$

The denominator polynomial B evaluates to zero at a pole. That is, if there is a pole at location $z = p$ (a complex number), then $B(p) = 0$. Assuming that $A(p) \neq 0$, then $\hat{X}(p)$ is not finite. Thus, the region of convergence cannot include any pole p that is not cancelled by a zero. This fact, combined with the fact that a causal signal always has a RoC of the form of the left one in figure 12.2, leads to the following simple **stability criterion for causal systems**:

A discrete-time causal system is stable if and only if all the poles of its transfer function lie inside the unit circle.

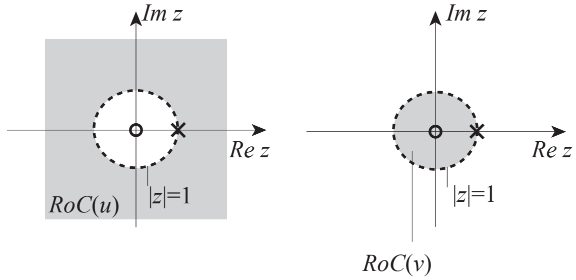

Figure 12.3: Pole-zero plots for the unit step u and its anti-causal cousin v. The regions of convergence are the shaded area in the complex plane, not including the unit circle. Both Z tranforms, $\hat{U}$ and $\hat{V}$, have one pole at $z = 1$ and one zero at $z = 0$.

A subtle fact about rational Z transforms is that the region of convergence is always bordered by the pole locations. This is evident in figure 12.3, where the single pole at $z = 1$ lies on the boundary of the two possible regions of convergence. In fact, the rational polynomial

$$\frac{z}{z-1}$$

can be associated with only three possible Z transforms, two of which have the two regions of convergence shown in figure 12.3, plus the one not shown where $RoC = \emptyset$.

Although a polynomial of order N has N roots, these roots are not necessarily distinct. Consider the (rather trivial) polynomial

$$A(z) = z^2.$$

This has order 2, and hence two roots, but both roots are at $z = 0$. Consider a Z transform given by

$$\forall\, z \in RoC(x), \quad \hat{X}(z) = \frac{z^2}{(z-1)^2}.$$

This has two zeros at $z = 0$, and two poles at $z = 1$. We say that the zero at $z = 0$ has **multiplicity** two. Similarly, the pole at $z = 1$ has multiplicity two. This multiplicity is indicated in a pole-zero plot by a number adjacent to the pole or zero, as shown in figure 12.4.

Example 12.12: Consider a signal x that is equal to the delayed Kronecker delta function,

$$\forall\, n \in Integers, \quad x(n) = \delta(n - M),$$

where $M \in Integers$ is a constant. Its Z transform is easy to find using the sifting rule,

$$\forall\, z \in RoC(x), \quad \hat{X}(z) = \sum_{m=-\infty}^{\infty} \delta(m - M) z^{-m} = z^{-M} = 1/z^M.$$

If $M > 0$, then this converges absolutely for any $z \neq 0$. Thus, if $M > 0$,

$$RoC(x) = \{z \in Complex \mid z \neq 0\}.$$

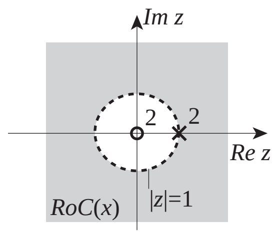

Figure 12.4: Poles and zeros with multiplicity greater than one are indicated by a number next to the cross or circle.

This Z transform has M poles at $z = 0$. Notice that this region of convergence, appropriately, has the form of that of a causal signal, figure 12.2(a), but where the circle has radius zero.

If $M < 0$, then the region of convergence is the entire set *Complex*, and the Z transform has M zeros at $z = 0$. This signal is anti-causal, and its *RoC* matches the structure of 12.2(c), where the radius of the circle is infinite. Note that this Z transform does not converge at infinity, which it would have to do if the signal were causal.

If $M = 0$, then $\hat{X}(z) = 1$ for all $z \in Complex$, so $RoC = Complex$, and there are no poles or zeros. This is a particularly simple Z transform.

Recall that for a causal signal, the Z transform must converge as $z \to \infty$. The region of convergence must include everything outside some circle, including infinity.[1] This implies that for a causal signal with a rational Z transform, the Z transform must be **proper**. A rational polynomial is proper when the order of the numerator is smaller than or equal to the order of the denominator. For example, if $M = -1$ in the previous example, then $x(n) = \delta(n+1)$ and $\hat{H}(z) = z$, which has numerator order one and denominator order zero. It is not proper, and indeed, it does not converge as $z \to \infty$. Any rational polynomial that has a denominator of higher order than the numerator will not converge as z goes to infinity, and hence cannot be the Z transform of a causal signal.

In the following chapter, table 13.1 gives many common Z tranforms, all of which are rational polynomials. Together with the properties discussed in the that chapter, we can find the Z transforms of many signals.

12.3 The Laplace transform

Consider a continuous-time signal x that is not absolutely integrable. Consider the scaled signal x_σ given by[2]

$$\forall\, t \in Reals, \quad x_\sigma(t) = x(t)e^{-\sigma t}, \tag{12.14}$$

[1] Some texts consider poles and zeros at infinity, in which case a causal signal cannot have a pole at infinity.

[2] The reason that this is different from the scaling by r^{-n} used to get the Z transform is somewhat subtle. The two methods are essentially equivalent, if we let $r = e^{\sigma}$. But scaling by $e^{-\sigma t}$ turns out to be more convenient for continuous-time systems, as we will see.

for some real number σ. Often, this signal is absolutely integrable when σ is chosen appropriately. This new signal, therefore, will have a CTFT, even if x does not.

Example 12.13: Consider the impulse response of the simplified helicopter system described in example 12.2. The output as a function of the input is given by

$$\forall\, t \in Reals, \quad y(t) = \frac{1}{M}\int_0^t x(\tau)d\tau.$$

The impulse response is found by letting the input be a Dirac delta function and using the sifting rule to get

$$\forall\, t \in Reals, \quad h(t) = u(t)/M,$$

where u is the continuous-time **unit step** in (12.4). This is not absolutely integrable, so this system is not stable. However, the scaled signal

$$\forall\, t \in Reals, \quad h_\sigma(t) = h(t)e^{-\sigma t}$$

is absolutely integrable if $\sigma > 0$. Its CTFT is

$$\begin{aligned}
\forall\, \sigma > 0, \forall \omega \in Reals, \quad H_\sigma(\omega) &= \int_{-\infty}^{\infty} h(t)e^{-\sigma t}e^{-i\omega t}dt \\
&= \frac{1}{M}\int_0^{\infty} e^{-\sigma t}e^{-i\omega t}dt \\
&= \frac{1}{M}\int_0^{\infty} e^{-(\sigma+i\omega)t}dt \\
&= \frac{1}{M(\sigma+i\omega)}\ .
\end{aligned}$$

The last step in example 12.13 uses the following useful fact from calculus,

$$\int_a^b e^{ct}dt = \frac{1}{c}(e^{cb} - e^{ca})\ , \tag{12.15}$$

for any $c \in Complex$ and $a, b \in Reals \cup \{-\infty, \infty\}$ where e^{cb} and e^{ca} are finite.

In general, the CTFT of the scaled signal x_σ in (12.14) is

$$\forall\, \omega \in Reals, \quad X_\sigma(\omega) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+i\omega)t}dt.$$

Notice that this is a function not just of ω, but also of σ. We are only sure it is valid for values of σ that yield an absolutely integrable signal h_σ.

Define the complex number

$$s = \sigma + i\omega.$$

Then we can write this CTFT as

$$\boxed{\forall\, s \in RoC(x), \quad \hat{X}(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt,} \qquad (12.16)$$

where $\hat{X}$ is a function called the **Laplace transform** of x,

$$\hat{X}\colon RoC(x) \to Complex$$

where $RoC(x) \subset Complex$ is given by

$$\boxed{RoC(x) = \{s = \sigma + i\omega \in Complex \mid x(t)e^{-\sigma t} \text{ is absolutely integrable.}\}} \qquad (12.17)$$

The Laplace tranform $\hat{H}$ of the impulse response h of an LTI system is called the **transfer function** of the system, just as with discrete-time systems.

> **Example 12.14:** Continuing example 12.13, we can recognize from the form of $H_\sigma(\omega)$ that the transfer function of the helicopter system is
>
> $$\forall\, s \in RoC(h), \quad \hat{H}(s) = \frac{1}{Ms}.$$
>
> The RoC is
>
> $$RoC(h) = \{s = \sigma + i\omega \in Complex \mid \sigma < 0\}$$

12.3.1 Structure of the region of convergence

As with the Z transform, the region of convergence is an essential part of a Laplace transform. It gives the domain of the function $\hat{X}$. Whether a complex number s is in the RoC depends only on σ, not on ω, as is evident in the definition (12.17). Since $s = \sigma + i\omega$, whether a complex number is in the region of convergence depends only on its real part. Once again, there are only three possible patterns for the region of convergence, shown in figure 12.5. Each figure illustrates the complex plane, and the shaded area is a region of convergence. Each possibility has vertical symmetry, in that whether a point is in the RoC depends only on its real part.

Figure 12.5(a) shows the RoC of a causal or right-sided signal. A continuous-time signal x is **right-sided** if $x(t) = 0$ for all $t < T$ for some $T \in Reals$. The RoC is the set of complex numbers $s = \sigma + i\omega$ where following integral converges:

$$\int_{-\infty}^{\infty} |x(t)e^{-\sigma t}|dt.$$

But if x is right-sided, then

$$\int_{-\infty}^{\infty} |x(t)e^{-\sigma t}|dt = \int_{T}^{\infty} |x(t)e^{-\sigma t}|dt.$$

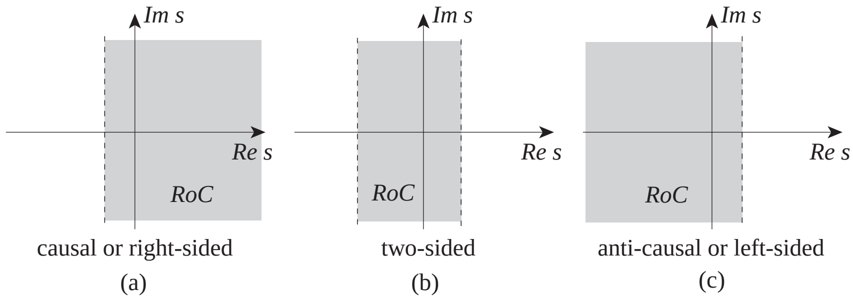

Figure 12.5: Three possible structures for the region of convergence of a Laplace transform.

If $T \geq 0$ and this integral converges for some given σ, then it must also converge for any $\tilde{\sigma} > \sigma$ because for all $t \geq 0$, $e^{-\tilde{\sigma}t} < e^{-\sigma t}$. Thus, if $s = \sigma + i\omega \in RoC(x)$, then the $RoC(x)$ must include all points in the complex plane on the vertical line passing through s and every point to the right of that line.[3]

If $T < 0$, then

$$\int_T^\infty |x(t)e^{-\sigma t}|dt = \int_T^0 |x(t)e^{-\sigma t}|dt + \int_0^\infty |x(t)e^{-\sigma t}|dt,$$

then the finite integral exists and is finite for all σ, so the same argument applies.

Figure 12.5(c) shows the *RoC* of a left-sided signal. A continuous-time signal x is **left-sided** if $x(t) = 0$ for all $t > T$ for some $T \in Reals$. By a similar argument, if $s = \sigma + i\omega \in RoC(x)$, then the $RoC(x)$ must include all points in the complex plane on the vertical line passing through s and every point to the left of that line.

Figure 12.5(b) shows the *RoC* of a signal that is a **two-sided signal**. Such a signal can always be expressed as a sum of a right-sided signal and left-sided signal. The *RoC* is the intersection of the regions of convergence for these two components.

Example 12.15: Using the same methods as in examples 12.13 and 12.14 we can find the Laplace transform of the continuous-time **unit step** signal u, given by

$$\forall\, t \in Reals, \quad u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}. \qquad (12.18)$$

The Laplace transform is

$$\forall\, s \in RoC(u), \quad \hat{U}(s) \quad = \quad \int_{-\infty}^{\infty} u(t)e^{-st}dt$$

[3] It is convenient but coincidental that the region of convergence is the right half of a plane when the sequence is right sided.

$$= \int_0^\infty e^{-st} dt$$

$$= \frac{1}{s},$$

where again we have used (12.15). The domain of $\hat{U}$ is

$$RoC(u) = \{s \in Complex \mid Re\{s\} > 0\}.$$

This region of convergence has the structure of figure 12.5(a), where the dashed line sits exactly on the imaginary axis. The region of convergence, therefore, is simply the right half of the complex plane.

Example 12.16: The signal v given by

$$\forall\, t \in Reals, \quad v(t) = -u(-t) = \begin{cases} -1, & t < 0 \\ 0, & t \geq 0 \end{cases},$$

has Laplace transform

$$\begin{aligned} \forall\, s \in RoC(v), \quad \hat{V}(s) &= \int_{-\infty}^{\infty} v(t) e^{-st} dt \\ &= -\int_{-\infty}^{0} e^{-st} dt \\ &= \frac{1}{s} \end{aligned}$$

with domain

$$RoC(v) = \{s \in Complex \mid Re\{s\} < 0\}.$$

This region of convergence has the structure of figure 12.5(c), where the dashed line coincides with the imaginary axis.

Notice that although the Laplace transforms $\hat{U}$ and $\hat{V}$ have the same algebraic form, namely, $1/s$, they are in fact different functions, because their domains are different.

Some signals have no meaningful Laplace transform.

Example 12.17: The signal x with $x(t) = 1$, for all $t \in Reals$, does not have a Laplace transform. We can write $x = u - v$, where u and v are defined in the previous examples. Thus, the region of convergence of x must be the intersection of the regions of convergence of u and v. However, these two regions have an empty intersection, so $RoC(x) = \emptyset$.

Viewed another way, the set $RoC(x)$ is the set of complex numbers s where

$$\int_{-\infty}^{\infty} |x(t)e^{-st}|dt = \int_{-\infty}^{\infty} |e^{-st}|dt < \infty.$$

But there is no such complex number s.

Note that the signal x in example 12.17 is periodic with any period $p \in Reals$ (because $x(t+p) = x(t)$ for any $p \in Reals$). Thus, it has a Fourier series representation. In fact, as shown in section 10.6.3, a periodic signal also has a Fourier transform representation, as long as we are willing to allow Dirac delta functions in the Fourier transform. (Recall that this means that there are values of ω where $X(\omega)$ will not be finite.) In the continuous-time case as in the discrete-time case, with periodic signals, the Fourier series is by far the simplest frequency-domain tool to use. The Fourier transform can also be used if we allow Dirac delta functions. The Laplace transform, however, is more problematic, because the region of convergence is empty.

12.3.2 Stability and the Laplace transform

If a continuous-time signal x is absolutely integrable, then it has a CTFT X that is finite for all $\omega \in Reals$. Moreover, the CTFT is equal to the Laplace transform evaluated on the imaginary axis,

$$\boxed{\forall\, \omega \in Reals, \quad X(\omega) = \hat{X}(s)|_{s=i\omega} = \hat{X}(i\omega).}$$

The complex number $s = i\omega$ is pure imaginary, and therefore lies on the imaginary axis. Recall that an LTI system is stable if and only if its impulse response is absolutely integrable. Thus

> A continuous-time LTI system with impulse response h is stable if an only if the transfer function $\hat{H}$, which is the Laplace transform of h, has a region of convergence that includes the imaginary axis.

Example 12.18: Consider the exponential signal h given by

$$\forall\, t \in Reals, \quad h(t) = e^{-at}u(t),$$

for some real or complex number a, where, as usual, u is the unit step. The Laplace transform is

$$\begin{aligned} \forall\, s \in RoC(h), \quad \hat{H}(s) &= \int_{-\infty}^{\infty} h(t)e^{-st}dt \\ &= \int_{0}^{\infty} e^{-at}e^{-st}dt \end{aligned}$$

$$= \int_0^\infty e^{-(s+a)t} dt$$
$$= \frac{1}{s+a},$$

where again we have used (12.15). It is evident from (12.15) that for this integral to be valid, the domain of $\hat{H}$ must be

$$RoC(h) = \{s \in Complex \mid Re\{s\} > -Re\{a\}\}.$$

This region of convergence has the structure of figure12.5(a), where the vertical dashed line passes through a.

Now suppose that h is the impulse response of an LTI system. That LTI system is stable if an only if $Re\{a\} > 0$. Indeed, if $Re\{a\} < 0$, then the impulse response grows without bound, because e^{-at} grows without bound as t gets large.

12.3.3 Rational Laplace tranforms and poles and zeros

All of the Laplace transforms we have seen so far are **rational polynomials** in s. In practice, most Laplace transforms of interest can be written as the ratio of two finite order polynomials in s,

$$\hat{X}(s) = \frac{A(s)}{B(s)}.$$

An exception is illustrated in the following example.

Example 12.19: Consider a signal x that is equal to the delayed Dirac delta function,

$$\forall\, t \in Reals, \quad x(t) = \delta(t - \tau),$$

where $\tau \in Reals$ is a constant. Its Laplace transform is easy to find using the sifting rule,

$$\forall\, s \in RoC(x), \quad \hat{X}(s) = \int_{-\infty}^{\infty} \delta(t-\tau) e^{-st} dt = e^{-s\tau}.$$

This has no finite-order rational polynomial representation.

Unlike the discrete-time case, pure time delays turn out to be rather difficult to realize in many physical systems that are studied using Laplace transforms, so we need not be overly concerned with them. We focus henceforth on rational Laplace transforms.

For a rational Laplace transform, the **order** of the polynomial A or B is the power of the highest power of s. For the exponential of example 12.18, the numerator polynomial is $A(s) = 1$, a zero-order polynomial, and the denominator is $B(s) = s + a$, a first-order polynomial. As with the Z transform, the roots of the numerator polynomial are called the **zeros** of the Laplace transform, and the roots of the denominator polynomial are called the **poles**.

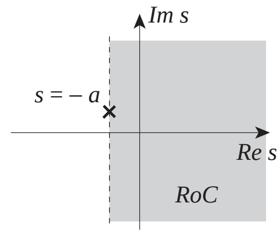

Figure 12.6: Pole-zero plot for the exponential signal of example 12.18, assuming a has a positive real part.

Example 12.20: The exponential of example 12.18 has a single pole at $s = -a$, and no zeros.[4] A pole-zero plot is shown in figure 12.6, where we assume that a is a complex number with a positive real part. The region of convergence includes the imaginary axis, so this signal is absolutely integrable.

As with Z transforms, the region of convergence of a rational Laplace transform bordered by the pole locations. Hence,

A continuous-time causal system is stable if and only if all the poles of its transfer function lie in the left half of the complex plane. That is, all the poles must have negative real parts.

Table 13.3 in the following chapter gives many common Laplace tranforms.

12.4 Summary

Many useful signals have no Fourier transform. A sufficient condition for a signal to have a Fourier transform that is finite at all frequencies is that the signal be absolutely summable (if it is a discrete-time signal) or absolutely integrable (if it is a continuous-time system).

Many useful systems are not stable, which means that even with a bounded input, the output may be unbounded. An LTI system is stable if and only if its impulse response is absolutely summable (discrete-time) or absolutely integrable (continuous-time).

Many signals that are not absolutely summable (integrable) can be scaled by an exponential to get a new signal that is absolutely summable (integrable). The DTFT (CTFT) of the scaled signal is called the Z transform (Laplace transform) of the signal.

[4] In some texts, it will be observed that as s approaches infinity, this Laplace transform approaches zero, and hence it will be said that there is a zero at infinity. So to avoid conflict with such texts, we might say that this Laplace transform has no finite zeros.

The Z transform (Laplace transform) is defined over a region of convergence, where the structure of the region of convergence depends on whether the signal is causal, anti-causal, or two-sided. The Z transform (Laplace transform) of the impulse response is called the transfer function of an LTI system. The region of convergence includes the unit circle (imaginary axis), if and only if the system is stable.

A rational Z transform (Laplace transform) has poles and zeros, and the poles bound the region of convergence. The locations of the poles and zeros yield considerable information about the system, including whether it is stable.

Exercises

Each problem is annotated with the letter **E, T, C** which stands for exercise, requires some thought, requires some conceptualization. Problems labeled **E** are usually mechanical, those labeled **T** require a plan of attack, those labeled **C** usually have more than one defensible answer.

1. **E** Consider the signal x given by

$$\forall\, n \in Integers, \quad x(n) = a^n u(-n),$$

   where a is a complex constant.

   (a) Find the Z transform of x. Be sure to give the region of convergence.

   (b) Where are the poles and zeros?

   (c) Under what conditions on a is x absolutely summable?

   (d) Assuming that x is absolutely summable, find its DTFT.

2. **T** Consider the signal x given by

$$\forall\, n \in Integers, \quad x(n) = \begin{cases} 1, & |n| \le M \\ 0, & \text{otherwise} \end{cases},$$

   for some integer $M > 0$.

   (a) Find the Z transform of x. Simplify so that there remain no summations. Be sure to give the region of convergence.

   (b) Where are the poles and zeros? Do not give poles and zeros that cancel each other out.

   (c) Under what conditions is x absolutely summable?

   (d) Assuming that x is absolutely summable, find its DTFT.

3. **T** Consider the **unit ramp** signal w given by

$$\forall\, n \in Integers, \quad w(n) = nu(n),$$

where u is the unit step. The following identity will be useful,

$$\sum_{m=0}^{\infty}(m+1)a^m = (\sum_{m=0}^{\infty} a^m)^2 = \frac{1}{(1-a)^2}\ . \qquad (12.19)$$

This is a generalization of the geometric series identity, given by (12.9). This series converges for any complex number a with $|a| < 1$, because

$$\begin{aligned}
\sum_{m=0}^{\infty}(m+1)|a|^m &= 1+2|a|+3|a|^2+\cdots \\
&= (1+|a|+|a|^2+\cdots)(1+|a|+|a|^2+\cdots) \\
&= (\sum_{m=0}^{\infty}|a|^m)^2 \\
&< \infty.
\end{aligned}$$

(a) Use the given identity to find the Z transform of the unit ramp. Be sure to give the region of convergence. Check your answer against that given on page 432.

(b) Sketch the pole-zero plot of the Z transform.

(c) Is the unit ramp absolutely summable?

4. **E** Sketch the pole-zero plots and regions of convergence for the Z transforms of the following impulse responses, and indicate whether a discrete-time LTI system with these impulse responses is stable:

(a) $h_1(n) = \delta(n) + 0.5\delta(n-1)$.

(b) $h_2(n) = (0.5)^n u(n)$.

(c) $h_3(n) = 2^n u(n)$.

5. **E** Consider the anti-causal continuous-time exponential signal x given by

$$\forall\, t \in Reals, \quad x(t) = -e^{-at}u(-t),$$

for some real or complex number a, where, as usual, u is the unit step.

(a) Show that the Laplace transform of x is

$$\hat{X}(s) = \frac{1}{s+a}$$

with region of convergence

$$RoC(x) = \{s \in Complex \mid Re\{s\} < -Re\{a\}\}.$$

(b) Where are the poles and zeros?

(c) Under what conditions on a is x absolutely integrable?

(d) Assuming that x is absolutely integrable, find its CTFT.

6. **E** This exercise demonstrates that the Laplace transform is linear. Show that if x and y are continuous-time signals, a and b are complex (or real) constants, and w is given by

$$\forall\, t \in Reals, \quad w(t) = ax(t) + by(t),$$

then the Laplace transform is

$$\forall\, s \in RoC(w), \quad \hat{W}(s) = a\hat{X}(s) + b\hat{Y}(s),$$

where

$$RoC(w) \supset RoC(x) \cap RoC(y).$$

7. **T** Let the causal sinusoidal signal y be given by

$$\forall\, t \in Reals, \quad y(t) = \cos(\omega_0 t)u(t),$$

where ω_0 is a real number and u is the unit step.

   (a) Show that the Laplace transform is

$$\forall\, s \in \{s \mid Re\{s\} > 0\}, \quad \hat{Y}(s) = \frac{s}{s^2 + \omega_0^2}\ .$$

   Hint: Use linearity, demonstrated in exercise 6, and Euler's relation.

   (b) Sketch the pole-zero plot and show the region of convergence.

8. **E** Consider a discrete-time LTI system with impulse response

$$\forall n, \quad h(n) = a^n \cos(\omega_0 n)u(n),$$

for some $\omega_0 \in Reals$. Determine for what values of a this system is stable.

9. **T** The continuous-time **unit ramp** signal w is given by

$$\forall t \in Reals, \quad x(t) = tu(t),$$

where u is the unit step.

   (a) Find the Laplace transform of the unit ramp, and give the region of convergence. **Hint**: Use integration by parts in (12.16) and the fact that $\int_0^\infty te^{-\sigma t}dt < \infty$ for $\sigma > 0$.

   (b) Sketch the pole-zero plot of the Laplace transform.

10. **E** Let h and g be the impulse response of two stable systems. They may be discrete-time or continuous-time. Let a and b be two complex numbers. Show that the system with impulse response $ah + bg$ is stable.

11. **T** Consider a series composition of two (continuous- or discrete-time) systems with impulse response h and g. The output v of the first system is related to its input x by $v = h * x$. The output y of the second system (and of the series composition) is $y = g * v$. Suppose both systems are stable. Show that the series composition is stable.
Hint: Use the definition of stability.

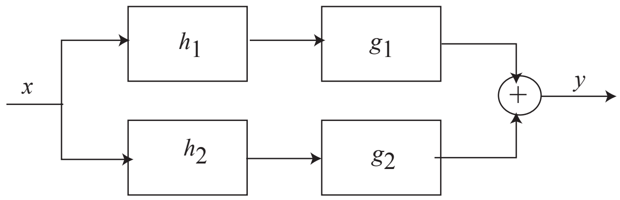

Figure 12.7: System composition for exercise 13.

12. **T** Let h be the impulse response of a stable discrete-time system, so it is absolutely summable, and denote

$$\|h\| = \sum_{n=-\infty}^{\infty} |h(n)|.$$

($\|h\|$ is called the **norm** of the impulse response.)

   (a) Suppose the input signal x is bounded by M, i.e. $\forall n, |x(n)| \le M$. Show that the output $y = h * x$ is bounded by $\|h\|M$.

   (b) Consider the input signal x where

$$\forall n \in Integers, \quad x(n) = \begin{cases} h(-n)/|h(-n)|, & h(n) \neq 0 \\ 0, & h(n) = 0. \end{cases}$$

   Show that $\|h\|$ is the smallest bound of the output $y = h * x$.

   (c) Let g be the impulse response of another stable system with norm $\|g\|$. Show that the norm satisfies the triangle inequality,

$$\|h + g\| \le \|h\| + \|g\|.$$

   (d) Suppose the two systems are placed in series. The composition has the impulse response $h * g$. Show that

$$\|h * g\| \le \|h\| \times \|g\|.$$

13. **E** Show that the series-parallel composition of figure 12.7 is stable if the four component systems are stable. Let h be the impulse response of the composition. Express h in terms of the component impulse responses and then estimate $\|h\|$ in terms of the norms of the components.

14. **E** Let x be a discrete-time signal of finite duration, i.e. $x(n) = 0$ for $n < M$ and $n > N$ where M and N are finite integers (positive or negative). Let $\hat{X}$ be its Z transform.

   (a) Show that all its poles (if any) are at $z = 0$.

   (b) Show that if x is causal it has N poles at $z = 0$.

15. **T** This problem relates the Z and Laplace transforms. Let x be a discrete-time signal with Z transform $\hat{X} : RoC(x) \to Complex$. Consider the continuous-time signal y related to x by

$$\forall t \in Reals, \quad y(t) = \sum_{n=-\infty}^{\infty} x(n)\delta(t - nT).$$

Here $T > 0$ is a fixed period. So y comprises delta functions located at $t = nT$ of magnitude $x(n)$.

(a) Use the sifting property and the definition (12.16) to find the Laplace transform $\hat{Y}$ of y. What is $RoC(y)$?

(b) Show that $\hat{Y}(s) = \hat{X}(e^{sT})$, where $\hat{X}(e^{sT})$ is $\hat{X}(z)$ evaluated at $s = e^{sT}$.

(c) Suppose $\hat{X}(z) = \frac{1}{z-1}$ with $RoC(x) = \{z \mid |z| > 1\}$. What are $\hat{Y}$ and $RoC(y)$?

Chapter 13

Laplace and Z Transforms

In the previous chapter, we defined Laplace and Z transforms to deal with signals that are not absolutely summable and systems that are not stable. The Z transform of the discrete-time signal x is given by

$$\forall\, z \in RoC(x), \quad \hat{X}(z) = \sum_{m=-\infty}^{\infty} x(m) z^{-m},$$

where $RoC(x)$ is the **region of convergence**, the region in which the sum above converges absolutely.

The Laplace transform of the continuous-time signal x is given by

$$\forall\, s \in RoC(x), \quad \hat{X}(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt,$$

where $RoC(x)$ is again the region of convergence, the region in which the integral above converges absolutely.

In this chapter, we explore key properties of the Z and Laplace transforms and give examples of transforms. We will also explain how, given a rational polynomial in z or s, plus a region of convergence, one can find the corresponding time-domain function. This **inverse transform** proves particularly useful, because compositions of LTI systems, studied in the next chapter, often lead to rather complicated rational polynomial descriptions of a transfer function.

Z transforms of common signals are given in table 13.1. Properties of the Z transform are summarized in table 13.2 and elaborated in the first section below.

13.1 Properties of the Z tranform

The Z transform has useful properties that are similar to those of the four Fourier transforms. They are summarized in table 13.2 and elaborated in this section.

Discrete-time signal $\forall\, n \in Integers$	**Z transform** $\forall\, z \in RoC(x)$	$\boldsymbol{Roc}(x) \subset \boldsymbol{Complex}$	**Reference**
$x(n) = \delta(n-M)$	$\hat{X}(z) = z^{-M}$	*Complex*	Example 12.12
$x(n) = u(n)$	$\hat{X}(z) = \frac{z}{z-1}$	$\{z \mid \lvert z\rvert > 1\}$	Example 12.7
$x(n) = a^n u(n)$	$\hat{X}(z) = \frac{z}{z-a}$	$\{z \mid \lvert z\rvert > \lvert a\rvert\}$	Example 13.3
$x(n) = a^n u(-n)$	$\hat{X}(z) = \frac{1}{1-a^{-1}z}$	$\{z \mid \lvert z\rvert < \lvert a\rvert\}$	Exercise 1 in chapter 12
$x(n) = \cos(\omega_0 n)u(n)$	$\hat{X}(z) = \frac{z^2 - z\cos(\omega_0)}{z^2 - 2z\cos(\omega_0)+1}$	$\{z \mid \lvert z\rvert > 1\}$	Example 13.3
$x(n) = \sin(\omega_0 n)u(n)$	$\hat{X}(z) = \frac{z\sin(\omega_0)}{z^2 - 2z\cos(\omega_0)+1}$,	$\{z \mid \lvert z\rvert > 1\}$	Exercise 1
$x(n) = \frac{1}{(N-1)!}(n-1)\cdots(n-N+1)\, a^{n-N}u(n-N)$	$\hat{X}(z) = \frac{1}{(z-a)^N}$	$\{z \mid \lvert z\rvert > \lvert a\rvert\}$	(13.13)
$x(n) = \frac{(-1)^N}{(N-1)!}(N-1-n)\cdots(1-n)\, a^{n-N}u(-n)$	$\hat{X}(z) = \frac{1}{(z-a)^N}$	$\{z \mid \lvert z\rvert < \lvert a\rvert\}$	(13.14)

Table 13.1: Z transforms of key signals. The signal u is the unit step (12.13), δ is the Kronecker delta, a is any complex constant, ω_0 is any real constant, M is any integer constant, and $N > 0$ is any integer constant.

Time domain $\forall\, n \in Integers$	**Frequency domain** $\forall\, z \in RoC$	***RoC***	**Name** (reference)
$w(n) = ax(n) + by(n)$	$\hat{W}(z) = a\hat{X}(z) + b\hat{Y}(z)$	$RoC(w) \supset RoC(x) \cap RoC(y)$	**Linearity** (section 13.1.1)
$y(n) = x(n-N)$	$\hat{Y}(z) = z^{-N}\hat{X}(z)$	$RoC(y) = RoC(x)$	**Delay** (section 13.1.2)
$y(n) = (x * h)(n)$	$\hat{Y}(z) = \hat{X}(z)\hat{H}(z)$	$RoC(y) \supset RoC(x) \cap RoC(h)$	**Convolution** (section 13.1.3)
$y(n) = x^*(n)$	$\hat{Y}(z) = [\hat{X}(z^*)]^*$	$RoC(y) = RoC(x)$	**Conjugation** (section 13.1.4)
$y(n) = x(-n)$	$\hat{Y}(z) = \hat{X}(z^{-1})$	$RoC(y) = \{z \mid z^{-1} \in RoC(x)\}$	**Time reversal** (section 13.1.5)
$y(n) = nx(n)$	$\hat{Y}(z) = -z\frac{d}{dz}\hat{X}(z)$	$RoC(y) = RoC(x)$	**Scaling by** n (page 432)
$y(n) = a^{-n}x(n)$	$\hat{Y}(z) = \hat{X}(az)$	$RoC(y) = \{z \mid az \in RoC(x)\}$	**Exponential scaling** (section 13.1.6)

Table 13.2: Properties of the Z transform. In this table, a, b are complex constants, and N is an integer constant.

13.1.1 Linearity

Suppose x and y have Z transforms $\hat{X}$ and $\hat{Y}$, that a, b are two complex constants, and that

$$w = ax + by.$$

Then the Z transform of w is

$$\forall\, z \in RoC(w), \quad \hat{W}(z) = a\hat{X}(z) + b\hat{Y}(z).$$

This follows immediately from the definition of the Z transform,

$$\begin{aligned} \hat{W}(z) &= \sum_{m=-\infty}^{\infty} w(m) z^{-m} \\ &= \sum_{m=-\infty}^{\infty} (ax(m) + by(m)) z^{-m} \\ &= a\hat{X}(z) + b\hat{Y}(z). \end{aligned}$$

The region of convergence of w must include at least the regions of convergence of x and y, since if $x(n)r^{-n}$ and $y(n)r^{-n}$ are absolutely summable, then certainly $(ax(n) + by(n))r^{-n}$ is absolutely summable. Conceivably, however, the region of convergence may be larger. Thus, all we can assert in general is

$$RoC(w) \supset RoC(x) \cap RoC(y). \tag{13.1}$$

Linearity is extremely useful because it makes it easy to find the Z transform of complicated signals that can be expressed a linear combination of signals with known Z transforms.

Example 13.1: We can use the results of example 12.12 plus linearity to find, for example, the Z transform of the signal x given by

$$\forall\, n \in Integers, \quad x(n) = \delta(n) + 0.9\delta(n-4) + 0.8\delta(n-5).$$

This is simply

$$\hat{X}(z) = 1 + 0.9z^{-4} + 0.8z^{-5}.$$

We can identify the poles by writing this as a rational polynomial in z (multiply top and bottom by z^5),

$$\hat{X}(z) = \frac{z^5 + 0.9z + 0.8}{z^5},$$

from which we see that there are 5 poles at $z = 0$. The signal is causal, so the region of convergence is the region outside the circle passing through the pole with the largest magnitude, or in this case,

$$RoC(x) = \{z \in Complex \mid z \neq 0\}.$$

Example 13.1 illustrates how to find the transfer function of any finite impulse response (FIR) filter. It also suggests that the transfer function of an FIR filter always has a region of convergence that includes the entire complex plane, except possibly $z = 0$. The region of convergence will also not include $z = \infty$ if the FIR filter is not causal.

Linearity can also be used to invert a Z transform. That is, given a rational polynomial and a region of convergence, we can find the time-domain function that has this Z transform. The general method for doing this will be considered in the next chapter, but for certain simple cases, we just have to recognize familiar Z transforms.

Example 13.2: Suppose we are given the Z transform

$$\forall\, z \in \{z \in Complex \mid z \neq 0\}, \quad \hat{X}(z) = \frac{z^5 + 0.9z + 0.8}{z^5}.$$

We can immediately recognize this as the Z transform of a causal signal, because it is a proper rational polynomial and the region of convergence includes the entire complex plane except $z = 0$ (thus, it has the form of figure 12.2(a)).

If we divide through by z^5, this becomes

$$\forall\, z \in \{z \in Complex \mid z \neq 0\}, \quad \hat{X}(z) = 1 + 0.9z^{-4} + 0.8z^{-5}.$$

By linearity, we can see that

$$\forall\, n \in Integers, \quad x(n) = x_1(n) + 0.9x_2(n) + 0.8x_3(n),$$

where x_1 has Z transform 1, x_2 has Z transform z^{-4}, and x_3 has Z transform z^{-5}. The regions of convergence for each Z transform must be at least that of x, or at least $\{z \in Complex \mid z \neq 0\}$. From example 12.12, we recognize these Z transforms, and hence obtain

$$\forall\, n \in Integers, \quad x(n) = \delta(n) + 0.9\delta(n-4) + 0.8\delta(n-5).$$

Another application of linearity uses Euler's relation to deal with sinusiodal signals.

Example 13.3: Consider the exponential signal x given by

$$\forall\, n \in Integers, \quad x(n) = a^n u(n),$$

where a is a complex constant. Its Z transform is

$$\hat{X}(z) = \sum_{m=-\infty}^{\infty} x(m)z^{-m} = \sum_{m=0}^{\infty} a^m z^{-m} = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}\,, \tag{13.2}$$

where we have used the geometric series identity (12.9). This has a zero at $z = 0$ and a pole at $z = a$. The region of convergence is

$$RoC(x) = \{z \in Complex \mid \sum_{m=0}^{\infty} |a|^m |z|^{-m} < \infty\} = \{z \mid |z| > |a|\}, \tag{13.3}$$

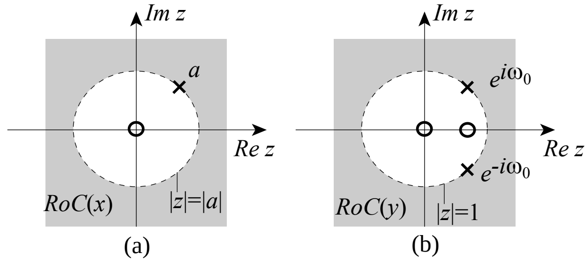

Figure 13.1: Pole-zero plots for the exponential signal x and the sinusoidal signal y of example 13.3.

the region of the complex plane outside the circle that passes through the pole. A pole-zero plot is shown in figure 13.1(a).

We can use this result plus linearity of the Z transform to determine the Z transform of the causal sinusoidal signal y given by

$$\forall\, n \in Integers, \quad y(n) = \cos(\omega_0 n)u(n).$$

Euler's relation implies that

$$y(n) = \frac{1}{2}\{e^{i\omega_0 n}u(n) + e^{-i\omega_0 n}u(n)\}.$$

Using (13.2) and linearity,

$$\begin{aligned}
\hat{Y}(z) &= \frac{1}{2}\left\{\frac{z}{z-e^{i\omega_0}} + \frac{z}{z-e^{-i\omega_0}}\right\} \\
&= \frac{1}{2}\frac{2z^2 - z(e^{i\omega_0}+e^{-i\omega_0})}{(z-e^{i\omega_0})(z-e^{-i\omega_0})} \\
&= \frac{z(z-\cos(\omega_0))}{z^2 - 2z\cos(\omega_0)+1}\,.
\end{aligned}$$

This has a zero at $z = 0$, another zero at $z = \cos(\omega_0)$, and two poles, one at $z = e^{i\omega_0}$ and the other at $z = e^{-i\omega_0}$. Both of these poles lie on the unit circle. A pole-zero plot is shown in figure 13.1(b), where we assume that $\omega_0 = \pi/4$. We know from (13.1) and (13.3) that the region of convergence is at least the area outside the unit circle. In this case, we can conclude that it is exactly the area outside the unit circle, because it must be bordered by the poles, and it must have the form of a region of convergence of a causal signal.

13.1.2 Delay

For any integer N (positive or negative) and signal x, let $y = D_N(x)$ be the signal given by

$$\forall n \in Integers, \quad y(n) = x(n-N).$$

Suppose x has Z transform $\hat{X}$ with domain $RoC(x)$. Then $RoC(y) = RoC(x)$ and

$$\forall z \in RoC(y), \quad \hat{Y}(z) = \sum_{m=-\infty}^{\infty} y(m) z^{-m} = \sum_{m=-\infty}^{\infty} x(m-N) z^{-m} = z^{-N} \hat{X}(z). \tag{13.4}$$

Thus

> If a signal is delayed by N samples, its Z transform is multiplied by z^{-N}.

13.1.3 Convolution

Suppose x and h have Z transforms $\hat{X}$ and $\hat{H}$. Let

$$y = x * h.$$

Then

$$\forall z \in RoC(y), \quad \hat{Y}(z) = \hat{X}(z)\hat{H}(z). \tag{13.5}$$

This follows from using the definition of convolution,

$$\forall n \in Integers, \quad y(n) = \sum_{m=-\infty}^{\infty} x(m) h(n-m),$$

in the definition of the Z transform,

$$\begin{aligned} \hat{Y}(z) &= \sum_{n=-\infty}^{\infty} y(n) z^{-n} = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x(m) z^{-m} h(n-m) z^{-(n-m)} \\ &= \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x(m) z^{-m} h(l) z^{-l} = \hat{X}(z)\hat{H}(z). \end{aligned}$$

The Z transform of y converges absolutely at least at values of z where both $\hat{X}$ and $\hat{H}$ converge absolutely. Thus,

$$RoC(y) \supset RoC(x) \cap RoC(h).$$

This is true because the double sum above can be written as

$$\sum_{n=-\infty}^{\infty} y(n) z^{-n} = \left(\sum_{m=-\infty}^{\infty} x(m) z^{-m} \right) \left(\sum_{l=-\infty}^{\infty} h(l) z^{-l} \right).$$

This obviously converges absolutely if each of the two factors converges absolutely. Note that the region of convergence may actually be larger than $RoC(x) \cap RoC(h)$. This can occur, for example, if the product (13.5) results in zeros of $\hat{X}(z)$ cancelling poles of $\hat{H}(z)$ (see exercise 3).

If h is the impulse response of an LTI system, then its Z transform is called the **transfer function** of the system. The result (13.5) tells us that the Z transform of the output is the product of the Z transform of the input and the transfer function. The transfer function, therefore, serves the same role as the frequency response. It converts convolution into simple multiplication.

13.1.4 Conjugation

Suppose x is a complex-valued signal. Let y be defined by

$$\forall\, n \in Integers, \quad y(n) = [x(n)]^*.$$

Then

$$\forall\, z \in RoC(y), \quad \hat{Y}(z) = [\hat{X}(z^*)]^*,$$

where

$$RoC(y) = RoC(x).$$

This follows because

$$\begin{aligned}
\forall\, z \in RoC(x), \quad \hat{Y}(z) &= \sum_{n=-\infty}^{\infty} y(n) z^{-n} \\
&= \sum_{n=-\infty}^{\infty} x^*(n) z^{-n} \\
&= \left[\sum_{n=-\infty}^{\infty} x(n)(z^*)^{-n} \right]^* \\
&= [\hat{X}(z^*)]^*.
\end{aligned}$$

If x happens to be a real signal, then $y = x$, so $\hat{Y} = \hat{X}$, so

$$\hat{X}(z) = [\hat{X}(z^*)]^*.$$

The key consequence is:

> For the Z transform of a real-valued signal, poles and zeros occur in complex-conjugate pairs. That is, if there is a zero at $z = q$, then there must be a zero at $z = q^*$, and if there is a pole at $z = p$, then there must be a pole at $z = p^*$.

This is because

$$0 = \hat{X}(q) = (\hat{X}(q^*))^*$$

Similarly, if there is a pole at $z = p$, then there must also be a pole at $z = p^*$.

Example 13.4: Example 13.3 gave the Z transform of a signal of the form $x(n) = a^n u(n)$, where a is allowed to be complex, and the Z tranform of a signal of the form $y(n) = \cos(\omega_0 n) u(n)$, which is real-valued. The pole-zero plots are shown in figure 13.1. In that figure, the complex signal has a pole at $z = a$, and none at $z = a^*$. But the real signal has a pole at $z = e^{i\omega_0}$ and a matching pole at the complex conjugate, $z = e^{-i\omega_0}$.

13.1.5 Time reversal

Suppose x has Z transform $\hat{X}$ and y is obtained from x by reversing time, so that

$$\forall\, n \in Integers, \quad y(n) = x(-n).$$

Then

$$\forall z \in \{z \in Complex \mid z^{-1} \in Roc(x)\}, \quad \hat{Y}(z) = \hat{X}(z^{-1}).$$

This is evident from the definition of the Z transform, which implies that

$$\hat{Y}(z) = \sum_{m=-\infty}^{\infty} x(-m)z^{-m} = \sum_{n=-\infty}^{\infty} x(n)(z^{-1})^{-n} = \hat{X}(z^{-1}),$$

where $\hat{X}(z^{-1})$ is $\hat{X}$ evaluated at z^{-1}.

13.1.6 Multiplication by an exponential

Suppose x has Z transform $\hat{X}$, a is a complex constant, and $y(n) = a^{-n}x(n)$ for all n. Then

$$\forall z \in \{z \in Complex \mid az \in RoC(x)\}, \quad \hat{Y}(z) = \hat{X}(az),$$

where $\hat{X}(az)$ is $\hat{X}$ evaluated at az. This is because

$$\hat{Y}(z) = \sum_{m=-\infty}^{\infty} y(m)z^{-m} = \sum_{m=-\infty}^{\infty} x(m)(az)^{-m} = \hat{X}(az).$$

Note that if $\hat{X}$ has a pole at p (or a zero at q), then $\hat{Y}$ has a pole at p/a (or a zero at q/a).

Example 13.5: Suppose x is given by

$$\forall\, n \in Integers, \quad x(n) = a^n u(n).$$

Then we know from example 13.3 that

$$\forall\, z \in \{z \mid |z| > |a|\}, \quad \hat{X}(z) = \frac{z}{z-a}.$$

This has a pole at $z = a$. Now let $y(n) = a^{-n}x(n) = u(n)$. The Z transform is

$$\hat{Y}(z) = \hat{X}(az) = \frac{az}{az-a} = \frac{z}{z-1},$$

as expected. Moreover, this has a pole at $z = a/a = 1$, as expected, and the region of convergence is indeed given by

$$\{z \in Complex \mid az \in RoC(x)\} = \{z \in Complex \mid |z| > 1\}.$$

Probing further: Derivatives of Z transforms

Calculus on complex-valued functions of complex variables can be somewhat intricate. Suppose $\hat{X}$ is a function of a complex variable. The derivative can be defined as a limit,

$$\frac{d}{dz}\hat{X}(z) = \lim_{\varepsilon \to 0} \frac{\hat{X}(z+\varepsilon) - \hat{X}(z)}{\varepsilon},$$

where ε is a complex variable that can approach zero from any direction in the complex plane. The derivative exists if the limit does not depend on the direction. If the derivative exists at all points within a distance $\varepsilon > 0$ of a point z in the complex plane, then $\hat{X}$ is said to be **analytic** at z. A Z transform is a series of the form

$$\forall\, z \in RoC(x), \quad \hat{X}(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}.$$

This is called a **Laurent series** in the theory of complex variables. It can be shown that a Laurent series is analytic at all points $z \in RoC(x)$, and that the derivative is

$$\forall\, z \in RoC(x), \quad \frac{d}{dz}\hat{X}(z) = \sum_{m=-\infty}^{\infty} -m x(m) z^{-m-1}.$$

We can use this fact to show that the Z transform of y given by $y(n) = nx(n)$ is

$$\forall z \in Roc(x), \quad \hat{Y}(z) = -z\frac{d}{dz}\hat{X}(z).$$

This is because

$$\hat{Y}(z) = \sum_{n=-\infty}^{\infty} nx(n)z^{-n} = \sum_{n=-\infty}^{\infty} (-z)\frac{d}{dz}x(n)z^{-n} = -z\frac{d}{dz}\hat{X}(z).$$

It is not difficult to show that $Roc(y) = Roc(x)$ (see exercise 5).

This property can be used to find other Z transforms. For example, the Z transform of the unit step, $x = u$, is $\hat{X}(z) = z/(z-1)$, with $RoC(x) = \{z \in Complex \mid |z| > 1\}$. So the Z transform of the **unit ramp** y, given by $y(n) = nu(n)$, is

$$\hat{Y}(z) = -z\frac{d}{dz}\frac{z}{z-1} = \frac{z}{(z-1)^2},$$

with $RoC(y) = \{z \in Complex \mid |z| > 1\}$. Another method for finding the Z transform of the unit ramp is given in exercise 3 of chapter 12.

13.1.7 Causal signals and the initial value theorem

Consider a causal discrete-time signal x. Its Z transform is

$$\forall\, z \in \{z \in Complex \mid |z| > r\}, \quad \hat{X}(z) = \sum_{m=0}^{\infty} x(m) z^{-m},$$

for some r (the largest magnitude of a pole). Then

$$\lim_{z\to\infty} \sum_{m=0}^{\infty} x(m) z^{-m} = x(0) + \lim_{z\to\infty} \sum_{m=1}^{\infty} x(m) z^{-m} = x(0).$$

This is because as z goes to ∞, each term $x(m)z^{-m}$ goes to zero. Thus

$$\boxed{\text{If } x \text{ is causal, } x(0) = \lim_{z\to\infty} \hat{X}(z)}.$$

This is called the **initial value theorem**.

> **Example 13.6:** The Z transform of the unit step $x(n) = u(n)$ is $\hat{X}(z) = z/(z-1)$, so, as expected,
>
> $$x(0) = \lim_{z\to\infty} \hat{X}(z) = \lim_{z\to\infty} \frac{z}{z-1} = \lim_{z\to\infty} \frac{1}{1-z^{-1}} = 1,$$
>
> because
>
> $$\lim_{z\to\infty} z^{-1} = 0.$$

Suppose a Z transform $\hat{X}$ is the rational polynomial

$$\hat{X}(z) = \frac{a_M z^M + a_{M-1} z^{M-1} \cdots + a_0}{z^N + b_{N-1} z^{N-1} + \cdots + b_0}\ .$$

If x is causal, then this rational polynomial must be **proper**. Were this not the case, if $M > N$, then by the initial value theorem, we would have

$$x(0) = \lim_{z\to\infty} \hat{X}(z) = \infty,$$

which is certainly not right.

> **Example 13.7:** Consider the Z transform
>
> $$\forall\, z \in Complex, \quad \hat{X}(z) = z.$$
>
> This is not a proper rational polynomial (the numerator has order 1 and the denominator, which is 1, has order 0). From example 12.12, we know that this corresponds to
>
> $$\forall\, n \in Integers, \quad x(n) = \delta(n+1).$$
>
> This is not a causal signal.

13.2 Frequency response and pole-zero plots

A pole-zero plot can be used to get a quick estimate of key properties of an LTI system. We have already seen that it reveals whether the system is stable. It also reveals key features of the frequency response, such as whether the system is highpass or lowpass.

Consider a stable discrete-time LTI system with impulse response h, frequency response H, and rational transfer function $\hat{H}$. We know that the frequency response and transfer function are related by

$$\forall \omega \in Reals, \quad H(\omega) = \hat{H}(e^{i\omega}).$$

That is, the frequency response is equal to the Z transform evaluated on the unit circle. The unit circle is in the region of convergence because the system is stable.

Assume that $\hat{H}$ is a rational polynomial, in which case we can express it in terms of the first-order factors of the numerator and denominator polynomials,

$$\hat{H}(z) = \frac{(z - q_1) \cdots (z - q_M)}{(z - p_1) \cdots (z - p_N)} ,$$

with zeros at $q_1, \cdots, q_M$ and poles at $p_1, \cdots, p_N$. The zeros and poles may be repeated (i.e., they may have multiplicity greater than one). The frequency response is therefore

$$\forall\ \omega \in Reals, \quad H(\omega) = \frac{(e^{i\omega} - q_1) \cdots (e^{i\omega} - q_M)}{(e^{i\omega} - p_1) \cdots (e^{i\omega} - p_N)} .$$

The magnitude response is

$$\forall\ \omega \in Reals, \quad |H(\omega)| = \frac{|e^{i\omega} - q_1| \cdots |e^{i\omega} - q_M|}{|e^{i\omega} - p_1| \cdots |e^{i\omega} - p_N|} .$$

Each of these factors has the form

$$|e^{i\omega} - b|$$

where b is the location of either a pole or a zero. The factor $|e^{i\omega} - b|$ is just the distance from $e^{i\omega}$ to b in the complex plane.

Of course, $e^{i\omega}$ is a point on the unit circle. If that point is close to a zero at location q, then the factor $|e^{i\omega} - q|$ is small, so the magnitude response will be small. If that point is close to a pole at p, then the factor $|e^{i\omega} - p|$ is small, but since this factor is in the denominator, the magnitude response will be large. Thus,

> The magnitude response of a stable LTI system may be estimated from the pole-zero plot of its transfer function. Starting at $\omega = 0$, trace counterclockwise around the unit circle as ω increases. If you pass near a zero, then the magnitude response should dip. If you pass near a pole, then the magnitude response should rise.

Example 13.8: Consider the causal LTI system of example 9.16, which is defined by the difference equation

$$\forall\, n \in Integers, \quad y(n) = x(n) + 0.9y(n-1).$$

We can find the transfer function by taking Z transforms on both sides, using linearity, to get

$$\hat{Y}(z) = \hat{X}(z) + 0.9z^{-1}\hat{Y}(z).$$

The transfer function is

$$\hat{H}(z) = \frac{\hat{Y}(z)}{\hat{X}(z)} = \frac{1}{1-0.9z^{-1}} = \frac{z}{z-0.9}.$$

This has a pole at $z = 0.9$, which is closest to $z = 1$ on the unit circle, and a zero at $z = 0$, which is equidistant from all points on the unit circle. The zero, therefore, has no effect on the magnitude response. The pole is closest to $z = 1$, which corresponds to $\omega = 0$, so the magnitude response peaks at $\omega = 0$, as shown in figure 9.12.

Example 13.9: Consider a legnth-4 moving average. Using methods like those in example 9.12, we can show that the transfer function is

$$\forall\, z \in \{z \in Complex \mid z \neq 0\}, \quad \hat{H}(z) = \frac{1}{4} \cdot \frac{1-z^{-4}}{1-z^{-1}} = \frac{1}{4}\frac{z^4-1}{z^3(z-1)}.$$

The numerator polynomial has roots at the four roots of unity, which are $z = 1$, $z = e^{i\pi/2}$, $z = -1$, and $z = e^{i3\pi/2}$. Thus, we can write this transfer function as

$$\begin{aligned} & \forall\, z \in \{z \in Complex \mid z \neq 0\}, \\ \hat{H}(z) &= \frac{1}{4}\frac{(z-1)(z-e^{i\pi/2})(z+1)(z-e^{i3\pi/2})}{z^3(z-1)} \\ &= \frac{1}{4}\frac{(z-e^{i\pi/2})(z+1)(z-e^{i3\pi/2})}{z^3}. \end{aligned}$$

The $(z-1)$ factors in the numerator and denominator cancel (fortunately, or we would have a pole at $z = 1$, on the unit circle, and we would have to conclude that the system was unstable). A pole-zero plot is shown in figure 13.2.

The magnitude response is shown in figure 9.8. Relating that figure to the pole-zero plot, we see that the frequency response peaks at $z = 1$, and as we move around the unit circle, we pass through zero at $\omega = \pi/2$, or $z = e^{j\pi/2}$, and again through zero at $\omega = \pi$. The magnitude response is periodic with period 2π, so the zero at $z = e^{3i\pi/2}$ is also a zero at $z = e^{-i\pi/2}$, corresponding to a frequency of $\omega = -\pi/2$.

Continuous-time signal $\forall\, t \in Reals$	**Laplace transform** $\forall\, s \in RoC(x)$	***Roc*(x)**	**Reference**
$x(t) = \delta(t-\tau)$	$\hat{X}(s) = e^{-s\tau}$	*Complex*	Exercise 12.19
$x(t) = u(t)$	$\hat{X}(s) = 1/s$	$\{s \in Complex \mid Re\{s\} > 0\}$	Example 12.15
$x(t) = e^{-at}u(t)$	$\hat{X}(s) = \frac{1}{s+a}$	$\{s \in Complex \mid Re\{s\} > -Re\{a\}\}$	Example 12.18
$x(t) = -e^{-at}u(-t)$	$\hat{X}(s) = \frac{1}{s+a}$	$\{s \in Complex \mid Re\{s\} < -Re\{a\}\}$	Exercise 5
$x(t) = \cos(\omega_0 t)u(t)$	$\hat{X}(s) = \frac{s}{s^2+\omega_0^2}$	$\{s \mid Re\{s\} > 0\}$	Exercise 7
$x(t) = \sin(\omega_0 t)u(t)$	$\hat{X}(s) = \frac{\omega_0}{s^2+\omega_0^2}$	$\{s \mid Re\{s\} > 0\}$	Example 13.10
$x(t) = \frac{t^{N-1}}{(N-1)!}e^{-at}u(t)$	$\hat{X}(z) = \frac{1}{(s+a)^N}$	$\{s \in Complex \mid Re\{s\} > -Re\{a\}\}$	—
$x(t) = -\frac{t^{N-1}}{(N-1)!}e^{-at}u(-t)$	$\hat{X}(z) = \frac{1}{(s+a)^N}$	$\{s \in Complex \mid Re\{s\} < -Re\{a\}\}$	—

Table 13.3: Laplace transforms of key signals. The signal u is the unit step (12.18), δ is the Dirac delta, a is any complex constant, ω_0 is any real constant, τ is any real constant, and N is a positive integer.

Time domain $\forall\, t \in Reals$	s **domain** $\forall\, s \in RoC$	***RoC***	**Name** (reference)
$w(t) = ax(t) + by(t)$	$\hat{W}(s) = a\hat{X}(s) + b\hat{Y}(s)$	$RoC(w) \supset RoC(x) \cap RoC(y)$	**Linearity** (exercise 6)
$y(t) = x(t-\tau)$	$\hat{Y}(s) = e^{-s\tau}\hat{X}(s)$	$RoC(y) = RoC(x)$	**Delay** (exercise 7)
$y(t) = (x * h)(r)$	$\hat{Y}(s) = \hat{X}(s)\hat{H}(s)$	$RoC(y) \supset RoC(x) \cap RoC(h)$	**Convolution** (exercise 8)
$y(t) = x^*(t)$	$\hat{Y}(s) = [\hat{X}(s^*)]^*$	$RoC(y) = RoC(x)$	**Conjugation** (exercise 9)
$y(t) = x(ct)$	$\hat{Y}(s) = \hat{X}(s/c)/\lvert c\rvert$	$RoC(y) = \{s \mid s/c \in RoC(x)\}$	**Time scaling** (exercise 10)
$y(t) = tx(t)$	$\hat{Y}(s) = -\frac{d}{ds}\hat{X}(s)$	$RoC(y) = RoC(x)$	**Scaling by** t —
$y(t) = e^{at}x(t)$	$\hat{Y}(s) = \hat{X}(s-a)$	$RoC(y) = \{s \mid s-a \in RoC(x)\}$	**Exponential scaling** (exercise 11)
$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$	$\hat{Y}(s) = \hat{X}(s)/s$	$RoC(y) \supset RoC(x) \cap \{s \mid Re\{s\} > 0\}$	**Integration** (section 13.3.1)
$y(t) = \frac{d}{dt}x(t)$	$\hat{Y}(s) = s\hat{X}(s)$	$RoC(y) \supset RoC(x)$	**Differentiation** (page 44)

Table 13.4: Properties of the Laplace transform. In this table, a, b are complex constants, c and τ are real constants.

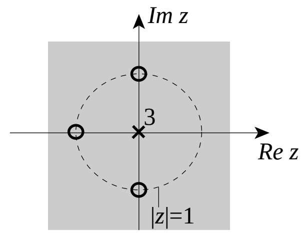

Figure 13.2: Pole-zero plot for a length-4 moving average system.

13.3 Properties of the Laplace transform

The Laplace transform has useful properties that are similar to those of the Z transform. They are summarized in table 13.4 and elaborated mostly in the exercises at the end of this chapter. In this section, we elaborate on one of the properties that is not shared by the Z transform, namely integration, and then use the properties to develop some examples. Key Laplace transforms are given in table 13.3.

13.3.1 Integration

Let y be defined by

$$\forall\, t \in Reals, \quad y(t) = \int_{-\infty}^{t} x(\tau)d\tau.$$

The Laplace tranform is

$$\forall\, s \in RoC(y), \quad \hat{Y}(s) = \hat{X}(s)/s,$$

where

$$RoC(y) \supset RoC(x) \cap \{s \mid Re\{s\} > 0\}.$$

This follows from the convolution property in table 13.4. We recognize that

$$y(t) = (x * u)(t),$$

where u is the unit step. Hence, from the convolution property,

$$\hat{Y}(s) = \hat{X}(s)\hat{U}(s)$$

and

$$RoC(y) \supset RoC(x) \cap RoC(u).$$

$\hat{U}$ and $RoC(u)$ are given in example 12.15, from which the property follows.

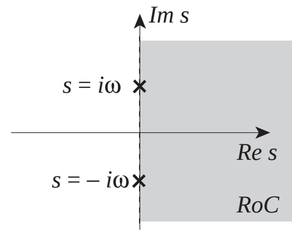

Figure 13.3: Pole-zero plot for the sinusoidal signal y of example 13.10.

13.3.2 Sinusoidal signals

Sinusoidal signals have Laplace transforms with poles on the imaginary axis, as illustrated in the following example.

Example 13.10: Let the causal sinusoidal signal y be given by

$$\forall\, t \in Reals, \quad y(t) = \sin(\omega_0 t)u(t),$$

where ω_0 is a real number and u is the unit step. Euler's relation implies that

$$y(t) = \frac{1}{2i}[e^{i\omega_0 t}u(t) - e^{-i\omega_0 t}u(t)].$$

Using (12.18) and linearity,

$$\begin{aligned} \hat{Y}(s) &= \frac{1}{2i}\left\{\frac{1}{s+i\omega_0} - \frac{1}{s-i\omega_0}\right\} \\ &= \frac{\omega_0}{s^2+\omega_0^2}\,. \end{aligned}$$

This has no finite zeros and two poles, one at $s = i\omega_0$ and the other at $s = -i\omega_0$. Both of these poles lie on the imaginary axis, as shown in figure 13.3. The region of convergence is the right half of the complex plane. Note that if this were the impulse response of an LTI system, that system would not be stable. The region of convergence does not include the imaginary axis.

13.3.3 Differential equations

We can use the differentiation property in table 13.4 to solve differential equations with constant coefficients.

Example 13.11: **??** In the tuning fork example of example 2.16, the displacement y of a tine is related to the acceleration of the tine by

$$\ddot{y}(t) = -\omega_0^2 y(t),$$

where ω_0 is a real constant. Let us assume that the tuning fork is initially at rest, and an external input x (representing say, a hammer strike) adds to the acceleration as follows,

$$\ddot{y}(t) = -\omega_0^2 y(t) + x(t).$$

We can use Laplace transforms to find the impulse response of this LTI system. Taking Laplace transforms on both sides, using linearity and the differentiation property,

$$\forall\, s \in RoC(y) \cap RoC(x), \quad s^2 \hat{Y}(s) = -\omega_0^2 \hat{Y}(s) + \hat{X}(s).$$

From this, we can find the transfer function of the system,

$$\hat{H}(s) = \frac{\hat{Y}(s)}{\hat{X}(s)} = \frac{1}{s^2 + \omega_0^2}\ .$$

Comparing this with example 13.10, we see that this differs only by a scaling by ω_0 from the Laplace transform in that example. Thus, the pole-zero plot of the tuning fork is shown in figure 13.3, and the impulse response is given by

$$\forall\, t \in Reals, \quad h(t) = \sin(\omega_0 t) u(t) / \omega_0.$$

Interestingly, this implies that the tuning fork is not stable. This impulse response is not absolutely integrable. However, this model of the tuning fork is idealized. It fails to account for loss of energy due to friction. A more accurate model would be stable.

The above example can be easily generalized to find the transfer function of any LTI system described by a differential equation. In fact, Laplace transforms offer a powerful and effective way to solve differential equations.

In the previous example, we inverted the Laplace transform by recognizing that it matched the example before that. In the next chapter, we will give a more general method for inverting a Laplace transform.

13.4 Frequency response and pole-zero plots, continuous time

Just as with Z transforms, the pole-zero plot of a Laplace transform can be used to get a quick estimate of key properties of an LTI system. Consider a stable continuous-time LTI system with impulse response h, frequency response H, and rational transfer function $\hat{H}$. We know that the frequency response and transfer function are related by

$$\forall \omega \in Reals, \quad H(\omega) = \hat{H}(i\omega).$$

That is, the frequency response is equal to the Laplace transform evaluated on the imaginary axis. The imaginary axis is in the region of convergence because the system is stable.

Assume that $\hat{H}$ is a rational polynomial, in which case we can express it in terms of the first-order factors of the numerator and denominator polynomials,

$$\hat{H}(s) = \frac{(s - q_1) \cdots (s - q_M)}{(s - p_1) \cdots (s - p_N)}\ ,$$

with zeros at $q_1, \cdots, q_M$ and poles at $p_1, \cdots, p_N$. The zeros and poles may be repeated (i.e., they may have multiplicity greater than one). The frequency response is therefore

$$\forall\, \omega \in Reals, \quad H(\omega) = \frac{(i\omega - q_1)\cdots(i\omega - q_M)}{(i\omega - p_1)\cdots(i\omega - p_N)}\,.$$

The magnitude response is

$$\forall\, \omega \in Reals, \quad |H(\omega)| = \frac{|i\omega - q_1|\cdots|i\omega - q_M|}{|i\omega - p_1|\cdots|i\omega - p_N|}\,.$$

Each of these factors has the form

$$|i\omega - b|$$

where b is the location of either a pole or a zero. The factor $|i\omega - b|$ is just the distance from $i\omega$ to b in the complex plane.

Of course, $i\omega$ is a point on the imaginary axis. If that point is close to a zero at location q, then the factor $|i\omega - q|$ is small, so the magnitude response will be small. If that point is close to a pole at p, then the factor $|i\omega - p|$ is small, but since this factor is in the denominator, the magnitude response will be large. Thus,

> The magnitude response of a stable LTI system may be estimated from the pole-zero plot of its transfer function. Starting at $i\omega = 0$, trace upwards and downwards along the imaginary axis to increase or decrease ω. If you pass near a zero, then the magnitude response should dip. If you pass near a pole, then the magnitude response should rise.

Example 13.12: Consider an LTI system with transfer function given by

$$\forall\, s \in \{s \mid Re\{s\} > Re\{a\}\}, \quad H(s) = \frac{s}{(s-a)(s-a^*)}\,.$$

Suppose that $a = c + i\omega_0$. Figure 13.4 shows three pole-zero plots for $\omega_0 = 1$ and three values of c, namely $c = -1$, $c = -0.5$, and $c = -0.1$. The magnitude frequency responses can be calculated and plotted using the following Matlab code:

```
omega = [-10:0.05:10];
a1 = -1.0 + i;
H1 = i*omega./((i*omega - a1).*(i*omega-conj(a1)));
a2 = -0.5 + i;
H2 = i*omega./((i*omega - a2).*(i*omega-conj(a2)));
a3 = -0.1 + i;
H3 = i*omega./((i*omega - a3).*(i*omega-conj(a3)));
plot(omega, abs(H1), omega, abs(H2), omega, abs(H3));
```

The plots are shown together at the bottom of figure 13.4. The plot with the higher peaks corresponds to the pole-zero plot with the poles closer to the imaginary axis.

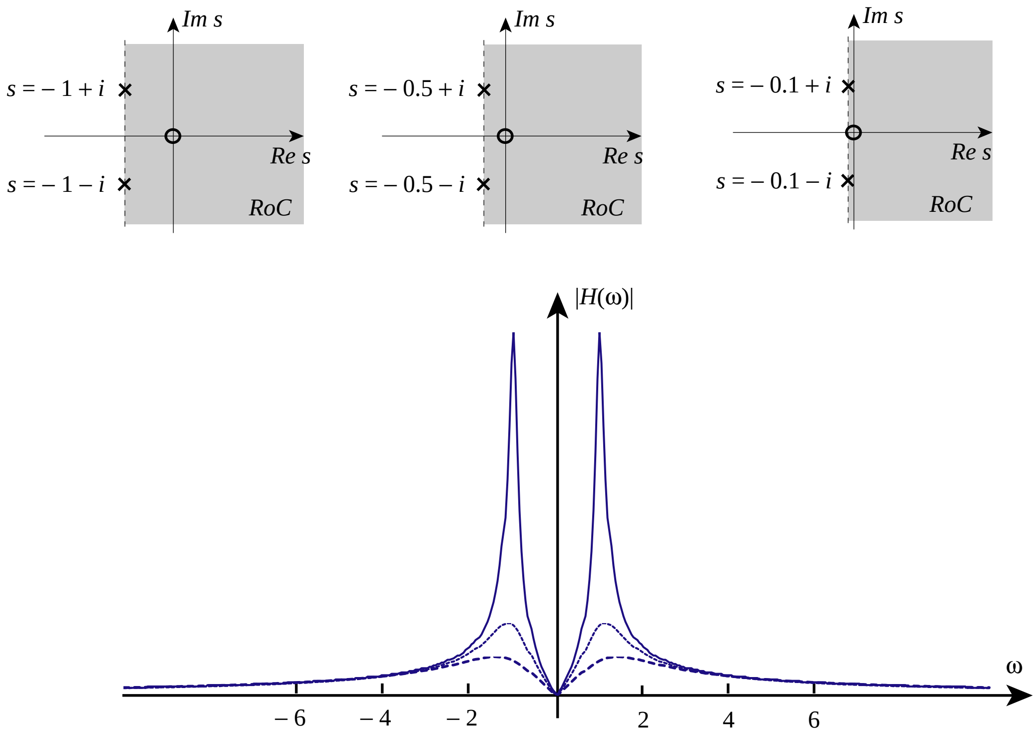

Figure 13.4: Pole-zero plots for the three transfer functions in example 13.12, and the three corresponding magnitude frequency responses.

13.5 The inverse transforms

There are two inverse transforms. The inverse Z transform recovers the discrete-time signal x from its Z transform $\hat{X}$. The inverse Laplace transform recovers the continuous-time signal x from its Laplace transform $\hat{X}$. We study the inverse Z transform in detail. The inverse Laplace transform is almost identical. The general approach is to break down a complicated rational polynomial into a sum of simple rational polynomials whose inverse transforms we recognize. We consider only the case where $\hat{X}$ can be expressed as a rational polynomial.

13.5.1 Inverse Z transform

The procedure is to construct the **partial fraction expansion** of $\hat{X}$, which breaks it down into a sum of simpler rational polynomials.

Example 13.13: Consider a Z transform given by

$$\forall\, z \in RoC(x), \quad \hat{X}(z) = \frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}\,. \tag{13.6}$$

This sum is called the **partial fraction expansion** of $\hat{X}$, and we will see below how to find it systematically. We can write this as

$$\forall\, z \in RoC(x), \quad \hat{X}(z) = \hat{X}_1(z) + \hat{X}_2(z),$$

where $\hat{X}_1(z) = -1/(z-1)$ and $\hat{X}_2(z) = 1/(z-2)$ are the two terms.

To determine the inverse Z transforms of the two terms, we need to know their regions of convergence. Recall from the linearity property that $RoC(x)$ includes the intersection of the regions of convergence of the two terms,

$$RoC(x) \supset RoC(x_1) \cap RoC(x_2). \tag{13.7}$$

Once we know these two regions of convergence, we can use table 13.1 to obtain the inverse Z transform of each term. By the linearity property the sum of these inverses is the inverse Z transform of $\hat{X}$.

$\hat{X}$ given by (13.6) has one pole at $z = 1$ and one pole at $z = 2$. From section 12.2.3 we know that $RoC(x)$ is bordered by these poles, so it has one of three forms:

1. $RoC(x) = \{z \in Complex \mid |z| < 1\}$,
2. $RoC(x) = \{z \in Complex \mid 1 < |z| < 2\}$, or
3. $RoC(x) = \{z \in Complex \mid |z| > 2\}$.

Suppose we have case (1), which implies that x is anti-causal. From (13.7), the region of convergence of the term $-1/(z-1)$ must be $\{z \in Complex \mid |z| < 1\}$. The only other possibility is $\{z \in Complex \mid |z| > 1\}$, which would violate (13.7) unless the intersection is empty (which would not be an interesting case). Thus, from table 13.1, the inverse Z transform of the first term must be the anti-causal signal $x_1(n) = u(-n)$, for all $n \in Integers$.

For the second term, $1/(z-2)$, its region of convergence could be either $\{z \in Complex \mid |z| < 2\}$ or $\{z \in Complex \mid |z| > 2\}$. Again, the second possibility would violate (13.7), so we must have the first possibility. This results in $x_2(n) = -2^{n-1}u(-n)$, from the last entry in table 13.1. Hence, the inverse Z transform is

$$\forall\, n \in Integers, \quad x(n) = u(-n) - 2^{n-1}u(-n).$$

If $RoC(x)$ is given by case (2), we rewrite (13.6) slightly as

$$\hat{X}(z) = -z^{-1}\frac{z}{z-1} + \frac{1}{z-2}.$$

The inverse Z transform of the first term is obtained from table 13.1, together with the delay property in table 13.2. The inverse Z transform of the second term is the same as in case (1). We conclude that in case (2) the inverse Z transform is the two-sided signal

$$\forall n, \quad x(n) = -u(n-1) - 2^{n-1}u(-n).$$

In case (3), we write (13.6) as

$$\hat{X}(z) = -z^{-1}\frac{z}{z-1} + z^{-1}\frac{z}{z-2},$$

and conclude that the inverse Z transform is the causal signal

$$\forall n, \quad x(n) = -u(n-1) + 2^{n-1}u(n-1).$$

We can generalize this example. Consider any **strictly proper** rational polynomial

$$\hat{X}(z) = \frac{A(z)}{B(z)} = \frac{a_M z^M + \cdots + a_1 z + a_0}{z^N + b_{N-1}z^{N-1} + \cdots + b_1 z + b_0}\ .$$

The numerator is of order M, the denominator is of order N. 'Strictly proper' means that $M < N$. We can factor the denominator,

$$\hat{X}(z) = \frac{a_M z^M + \cdots + a_1 z + a_0}{(z-p_1)^{m_1}(z-p_2)^{m_2}\cdots(z-p_k)^{m_k}}\ . \tag{13.8}$$

Thus $\hat{X}$ has k distinct poles at p_i, each with multiplicy m_i. Since the order of the denominator is N, it must be true that

$$N = \sum_{i=1}^{k} m_i\ . \tag{13.9}$$

The partial fraction expansion of (13.8) is

$$\hat{X}(z) = \sum_{i=1}^{k}\left[\frac{R_{i1}}{(z-p_i)} + \frac{R_{i2}}{(z-p_i)^2} + \cdots + \frac{R_{im_i}}{(z-p_i)^{m_i}}\right]\ . \tag{13.10}$$

A pole with multiplicity m_i contributes m_i terms to the partial fraction expansion, so the total number of terms is N, the order of the denominator, from (13.9). The coefficients R_{ij} are complex numbers called the **residues** of the pole p_i.

We assume that the poles $p_1, \cdots, p_N$ are indexed so that $|p_1| \leq \cdots |p_N|$. The $RoC(x)$ must have one of the following three forms:

1. $RoC = \{z \in Complex \mid |z| < |p_1|\}$,
2. $RoC = \{z \in Complex \mid |p_{j-1}| < |z| < |p_j|\}$, for $j \in \{2, \cdots, k\}$, or
3. $RoC = \{z \in Complex \mid |z| > |p_k|\}$.

As in example 13.13, each term in the partial fraction expansion has two possible regions of convergence, only one of which overlaps with $RoC(x)$. Thus, if we know $RoC(x)$, we can determine the region of convergence of each term of the partial fraction expansion, and then use table13.1 to find its inverse.

The following example illustrates how to find the residues.

Example 13.14: We will find the inverse Z transform of

$$\hat{X}(z) = \frac{2z+3}{(z-1)(z+2)} = \frac{R_1}{z-1} + \frac{R_2}{z+2}.$$

The residues R_1, R_2 can be found by matching coefficients on both sides. Rewrite the right-hand side as

$$\frac{(R_1+R_2)z + (2R_1 - R_2)}{(z-1)(z+2)} .$$

Matching the coefficients of the numerator polynomials on both sides we conclude that $R_1 + R_2 = 2$ and $2R_1 - R_2 = 3$. We can solve these simultaneous equations to determine that $R_1 = 5/3$ and $R_2 = 1/3$.

Alternatively, we can find residue R_1 by multiplying both sides by $(z-1)$ and evaluating at $z = 1$. That is,

$$R_1 = \left.\frac{2z+3}{z+2}\right|_{z=1} = \frac{5}{3}.$$

Similarly, we can find R_2 by we multiplying both sides by $z+2$ and evaluating at $z = -2$, to get

$$\frac{2z+3}{z-1}\,|_{z=-2} = R_2,$$

so $R_2 = 1/3$. Thus the partial fraction expansion is

$$\hat{X}(z) = \frac{5/3}{z-1} + \frac{1/3}{z+2} .$$

$RoC(x)$ is either

1. $\{z \in Complex \mid |z| < 1\}$,

2. $\{z \in Complex \mid 1 < |z| < 2\}$, or
3. $\{z \in Complex \mid |z| > 2\}$.

Knowing which case holds, we can find the inverse Z transform of each term from table 13.1. In the first case, x is the anti-causal signal

$$\forall n, \quad x(n) = -\frac{5}{3}u(-n) - \frac{1}{3}(-2)^{n-1}u(-n).$$

In the second case it is the two-sided signal

$$\forall n, \quad x(n) = \frac{5}{3}u(n-1) - \frac{1}{3}(-2)^{n-1}u(-n).$$

In the third case it is the causal signal

$$\forall n, \quad x(n) = \frac{5}{3}u(n-1) + \frac{1}{3}(-2)^{n-1}u(n-1).$$

If some pole of $\hat{X}$ has multiplicity greater than one, it is slightly more difficult to carry out the partial fraction expansion. The following example illustrates the method.

Example 13.15: Consider the expansion

$$\hat{X}(z) = \frac{2z+3}{(z-1)(z+2)^2} = \frac{R_1}{z-1} + \frac{R_{21}}{z+2} + \frac{R_{22}}{(z+2)^2}\ .$$

Again we can match coefficients and determine the residues. Alternatively, to obtain R_1 we multiply both sides by $(z-1)$ and evaluate the result at $z=1$, to get $R_1 = 5/9$. To obtain R_{22} we multiply both sides by $(z+2)^2$ and evaluate the result at $z=-2$, to get $R_{22} = 1/3$.

To obtain R_{21} we multiply both sides by $(z+2)^2$,

$$\frac{2z+3}{z-1} = \frac{(z+2)^2 R_1}{z-1} + R_{21}(z+2) + R_{22},$$

and then differentiate both sides with respect to z. We evaluate the result at $z=-2$, to get

$$\frac{d}{dz}\left.\frac{2z+3}{z-1}\right|_{z=-2} = R_{21}.$$

Hence $R_{21} = -5/9$. So the partial fraction expansion is

$$\frac{2z+3}{(z-1)(z+2)^2} = \frac{5/9}{z-1} - \frac{5/9}{z+2} + \frac{1/3}{(z+2)^2}.$$

Knowing the RoC, we can now obtain the inverse Z transform of $\hat{X}$. For instance, in the case where $RoC = \{z \in Complex \mid |z| > 2\}$, the inverse Z transform is the causal signal

$$\forall n, \quad x(n) = \frac{5}{9}u(n-1) - \frac{5}{9}(-2)^{n-1}u(n-1) + \frac{1}{3}(n-1)(-2)^{n-2}u(n-2).$$

In example 13.15, we used the next to the last entry in table 13.1 to find the inverse transform of the term $(1/3)/(z+2)^2$. That entry in the table is based on a generalization of the geometric series identity, given by (12.9). The first generalization is

$$\sum_{n=0}^{\infty}(n+1)a^n = (\sum_{n=0}^{\infty} a^n)^2 = \frac{1}{(1-a)^2} \ . \tag{13.11}$$

The series above converges for any complex number a with $|a| < 1$ (see exercise 3 of chapter 12). The broader generalization, for any integer $k \geq 1$, is

$$\frac{1}{k!}\sum_{n=0}^{\infty}(n+k)(n+k-1)\cdots(n+1)a^n = \frac{1}{(1-a)^{k+1}} \ , \tag{13.12}$$

for any complex number a with $|a| < 1$.

Consider then a Z transform $\hat{X}$ that has a pole at p of multiplicity m and no zeros. Since the pole p cannot belong to *RoC*, the *RoC* is either

$$\{z \in Complex \mid |z| > |p|\} \text{ or } \{z \in Complex \mid |z| < |p|\}.$$

In the first case we expand $\hat{X}$ in a series involving only the terms $z^{-n}, n \geq 0$,

$$\begin{aligned}
\hat{X}(z) &= \frac{1}{(z-p)^m} \\
&= \frac{z^{-m}}{(1-pz^{-1})^m} \\
&= z^{-m}\frac{1}{(m-1)!}\sum_{n=0}^{\infty}(m+n-1)\cdots(n+1)(pz^{-1})^n, \text{ using (13.12)} \\
&= \frac{1}{(m-1)!}\sum_{k=m}^{\infty}(k-1)\cdots(k-m+1)p^{k-m}z^{-k}, \text{ defining } k = n+m,
\end{aligned}$$

and the series converges for any z with $|z| > |p|$. We can match the coefficients of the powers of z in the Z transform definition,

$$\hat{X}(z) = \sum_{k=-\infty}^{\infty} x(k)z^{-k},$$

from which we can recognize that

$$\begin{aligned}
\forall k \in Integers, \quad x(k) &= \begin{cases} 0, & k < m \\ \frac{1}{(m-1)!}(k-1)\cdots(k-m+1)p^{k-m}, & k \geq m \end{cases} \\
&= \frac{1}{(m-1)!}(k-1)\cdots(k-m+1)p^{k-m}u(k-m).
\end{aligned} \tag{13.13}$$

In the second case, $RoC = \{z \in Complex \mid |z| < |p|\}$, we expand $\hat{X}$ in a series involving only the terms $z^{-n}, n \leq 0$,

$$\hat{X}(z) = \frac{1}{(z-p)^m}$$

$$
\begin{aligned}
&= \frac{1}{(-p)^m}\frac{1}{(1-p^{-1}z)^m} \\
&= \frac{1}{(-p)^m}\frac{1}{(m-1)!}\sum_{n=0}^{\infty}(m+n-1)\cdots(n+1)(p^{-1}z)^n, \text{ using (13.12)} \\
&= \frac{(-1)^m}{(m-1)!}\sum_{k=-\infty}^{0}(m-k-1)\cdots(1-k)p^{k-m}z^{-k}, \text{ defining } k=-n,
\end{aligned}
$$

and the series converges for any z with $|z| < |p|$. Again, we match powers of z in the Z transform definition to get

$$
\begin{aligned}
\forall k \in Integers, \quad x(k) &= \begin{cases} \frac{(-1)^m}{(m-1)!}(m-1-k)\cdots(1-k)p^{k-m}, & k \le 0 \\ 0, & k > 0 \end{cases} \\
&= \frac{(-1)^m}{(m-1)!}(m-1-k)\cdots(1-k)p^{k-m}u(-k). \qquad (13.14)
\end{aligned}
$$

Example 13.16: Suppose

$$\hat{X}(z) = \frac{1}{(z-2)^2}$$

with $RoC = \{z \in Complex \mid |z| > 2\}$. Then, by (13.13), $\hat{X}$ is the Z transform of the signal

$$\forall k \in Integers, \quad x(k) = \begin{cases} 0, & k < 2 \\ (k-1)2^{k-2}, & k \ge 2 \end{cases}.$$

Suppose

$$\hat{Y}(z) = \frac{1}{(z-2)^2}$$

with $RoC = \{z \in Complex \mid |z| < 2\}$. Then, by (13.14), $\hat{Y}$ is the Z transform of the signal

$$\forall k \in Integers, \quad y(k) = \begin{cases} (1-k)2^{k-2}, & k \le 0 \\ 0, & k > 0 \end{cases}.$$

Since the unit circle $\{z \in Complex \mid |z| = 1\} \subset RoC$, the DTFT of y is defined and given by

$$\forall \omega \in Reals, \quad Y(\omega) = \hat{Y}(e^{i\omega}) = \frac{1}{(e^{i\omega}-2)^2}.$$

Now that we know how to inverse transform all the terms of the partial fraction expansion, we can generalize the method used in example 13.15 to calculate the inverse Z transform of any $\hat{X}$ of the form

$$\hat{X}(z) = \frac{a_M z^M + \cdots + a_0}{z^N + b_{N-1}z^{N-1} + \cdots + b_0}.$$

Step 1 If $M \ge N$, divide through to obtain

$$\hat{X}(z) = c_{M-N}z^{M-N} + \cdots + c_0 + \hat{W}(z),$$

where $\hat{W}$ is strictly proper.

Step 2 Carry out the partial fraction expansion of $\hat{W}$ and, knowing the *RoC*, obtain the inverse Z transform w. Then from table 13.1,

$$\forall n, \quad x(n) = c_{m+l-N}\delta(n+m+l-N) + \cdots + c_0\delta(n) + w(n).$$

Example 13.17: We follow the procedure for

$$\hat{X}(z) = \frac{z^2 + z + 1 + z^{-1}}{(z+2)^2}.$$

First, to get this into the proper form, as a rational polynomial in z, notice that

$$\hat{X}(z) = z^{-1}\hat{Y}(z),$$

where

$$\hat{Y}(z) = \frac{z^3 + z^2 + z + 1}{(z+2)^2}.$$

Since z^{-1} corresponds to a one-step delay,

$$x(n) = y(n-1),$$

so if we find the inverse Z transform of $\hat{Y}$, then we have found the inverse Z trasform of $\hat{X}$.

Working now with $\hat{Y}$, step 1 yields

$$\hat{Y}(z) = z - 3 + \frac{9z + 13}{z^2 + 4z + 4}.$$

Step 2 gives

$$\hat{W}(z) = \frac{9z+13}{(z+2)^2} = \frac{-5}{(z+2)^2} + \frac{9}{z+2}\ .$$

Suppose $RoC = \{z \in Complex \mid |z| > 2\}$. Then from table 13.1,

$$\begin{aligned}
\forall n, \quad w(n) &= -5(n-1)(-2)^{n-2}u(n-2) + 9(-2)^{n-1}u(n-1), \\
\forall n, \quad y(n) &= \delta(n+1) - 3\delta(n) + w(n), \\
\forall n, \quad x(n) &= y(n-1).
\end{aligned}$$

Hence, for all $n \in Integers$,

$$x(n) = \delta(n) - 3\delta(n-1) - 5(n-2)(-2)^{n-3}u(n-3) + 9(-2)^{n-2}u(n-2).$$

13.5.2 Inverse Laplace transform

The procedure to calculate the inverse Laplace transform is virtually identical. Suppose the Laplace transform $\hat{X}$ is a rational polynomial

$$\hat{X}(s) = \frac{a_M s^M + \cdots + a_0}{s^N + b_{N-1}s^{N-1} + \cdots + b_0}.$$

We follow Steps 1 and 2 above. We divide through in case $M \geq N$ to obtain

$$\hat{X}(s) = c_{M-N}s^{M-N} + \cdots + c_0 + \hat{W}(s),$$

where $\hat{W}$ is strictly proper. We carry out the partial fraction expansion of $\hat{W}$. Knowing $RoC(x)$, we can again infer the region of convergence of each term. We then obtain the inverse Laplace transform term by term using table 13.3,

$$\forall t \in Reals, \quad x(t) = c_{m-n}\delta^{(M-N)}(t) + \cdots + c_0\delta(t) + w(t).$$

Here w is the inverse Laplace transform of $\hat{W}$, δ is the Dirac delta function, and $\delta^{(i)}$ is the ith derivative of the Dirac delta function.[1]

Example 13.18: We follow the procedure and obtain the partial fraction expansion of

$$\begin{aligned} \hat{X}(s) &= \frac{s^3+s^2+s+1}{s(s+2)^2} \\ &= 1 + \frac{-3s^2-3s+1}{s(s+2)^2} \\ &= 1 + \frac{1/4}{s} + \frac{-13/4}{s+2} + \frac{5/2}{(s+2)^2}. \end{aligned}$$

$\hat{X}$ has one pole at $s=0$ and a pole at $s=-2$ of multiplicity two. So its RoC has one of three forms:

1. $RoC = \{s \in Complex \mid Re\{s\} < -2\}$,
2. $RoC = \{s \in Complex \mid -2 < Re\{s\} < 0\}$, or
3. $RoC = \{s \in Complex \mid Re\{s\} > 0\}$.

We now use table 13.3 to obtain the inverse Laplace transform of each term. In case (1), the continuous-time signal is the anti-causal signal

$$\forall\, t, \quad x(t) = \delta(t) - \frac{1}{4}u(-t) + \frac{13}{4}\, e^{-2t}u(-t) - \frac{5}{2}\, te^{-2t}u(-t).$$

In case (2), it is the two-sided signal,

$$\forall t, \quad x(t) = \delta(t) - \frac{1}{4}u(-t) - \frac{13}{4}\, e^{-2t}u(t) + \frac{5}{2}\, te^{-2t}u(t).$$

In case (3), it is the causal signal,

$$\forall t, \quad x(t) = \delta(t) + \frac{1}{4}u(t) - \frac{13}{4}\, e^{-2t}u(t) + \frac{5}{2}\, te^{-2t}u(t).$$

[1] The derivative of δ is a function only in a formal sense, and we obtain its Laplace transform using the differentiation property in table 13.4.

Probing further: Inverse transform as an integral

Even if the Z transform is not a rational polynomial, we can recover the signal x from its Z transform, $\hat{X}: RoC(x) \to Complex$, using the DTFT. A non-empty $RoC(x)$ contains the circle of radius r for some $r > 0$. So the series in the equation

$$\hat{X}(re^{i\omega}) = \sum_{m=-\infty}^{\infty} x(m)(re^{i\omega})^{-m} = \sum_{m=-\infty}^{\infty} (x(m)r^{-m})e^{-i\omega m}$$

is absolutely summable. Hence the signal x_r: $\forall n, x_r(n) = x(n)r^{-n}$, has DTFT X_r: $\forall \omega, X_r(\omega) = \hat{X}(re^{i\omega})$. We can, therefore, obtain x_r as the inverse DTFT of X_r

$$\forall n, \quad x_r(n) = r^{-n}x(n) = \frac{1}{2\pi}\int_0^{2\pi} \hat{X}(re^{i\omega})e^{i\omega n}d\omega.$$

Multiplying both sides by r^n, we can recover x as

$$\forall n \in Integers, \quad x(n) = \frac{1}{2\pi}\int_0^{2\pi} \hat{X}(re^{i\omega})(re^{i\omega})^n d\omega. \tag{13.15}$$

This formula defines the inverse Z transform as an integral of the real variable ω. It is conventional to write the inverse Z transform differently. Express z as $z = re^{i\omega}$. Then as ω varies from 0 to 2π, z varies as

$$dz = re^{i\omega}id\omega = zid\omega, \text{ or } d\omega = \frac{dz}{iz}.$$

Substituting this in (13.15) gives,

$$\forall n, \quad x(n) = \frac{1}{2\pi}\oint \hat{X}(z)z^n \frac{dz}{iz} = \frac{1}{2\pi i}\oint \hat{X}(z)z^{n-1}dz.$$

Here the 'circle' in the integral sign, $\oint$, means that the integral in the complex z-plane is along any closed counterclockwise circle contained in $RoC(x)$. (An integral along a closed contour is called a **contour integral**.) In summary,

$$\boxed{\forall n \in Integers, \quad x(n) = \tfrac{1}{2\pi i}\oint \hat{X}(z)z^{n-1}dz,} \tag{13.16}$$

where the integral is along any closed counterclockwise circle inside $RoC(x)$.

We can similarly use the CTFT to recover any continuous-time signal x from its Laplace transform by

$$\boxed{\forall t \in Reals, \quad x(t) = \tfrac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \hat{X}(s)e^{st}ds}$$

where the integral is along any vertical line $(\sigma - i\infty, \sigma + i\infty)$ contained in $RoC(x)$.

Probing further: Differentiation property of the Laplace transform

We can use the inverse Laplace transform as given in the box on page451 to demonstrate the differentiation property in table 13.4. Let y be defined by

$$\forall\, t \in Reals, \quad y(t) = \frac{d}{dt}x(t).$$

We can write x in terms of its Laplace transform as

$$\forall t \in Reals, \quad x(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \hat{X}(s)e^{st}\,ds.$$

Differentiating this with respect to t is easy,

$$\forall t \in Reals, \quad \frac{d}{dt}x(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} s\hat{X}(s)e^{st}\,ds.$$

Consequently, $y(t) = dx(t)/dt$ is the inverse transform of $s\hat{X}(s)$, so

$$\forall\, s \in RoC(y), \quad \hat{Y}(s) = s\hat{X}(s),$$

where $RoC(y) \supset RoC(x)$.

13.6 Steady state response

Although it has been a fair amount of work, being able to compute an inverse transform for an arbitrary rational polynomial proves useful. Our first use will be to study the **stead-state response** of a causal and stable LTI system that has a sinusoidal input that starts at time zero.

If the input to an LTI system is a complex exponential,

$$\forall t \in Reals, \quad x(t) = e^{i\omega t},$$

then the output y is an exponential of the same frequency but with amplitude and phase given by $H(\omega)$,

$$\forall t \in Reals, \quad y(t) = H(\omega)e^{i\omega t},$$

where H is the frequency response. However, this result requires the exponential input to start at $t = -\infty$. In practice, of course, an input may start at some finite time, say at $t = 0$, but this result does not describe the output if the input is

$$\forall t \in Reals, \quad x(t) = e^{i\omega t}u(t). \tag{13.17}$$

We will see that if the system is stable and causal,[2] then the output y decomposes into two parts, a

[2]This result can be generalized to non-causal systems, but causal systems will be sufficient for our purposes.

transient output and a **steady state output**,

$$y = y_{tr} + y_{ss},$$

where the transient becomes vanishingly small for large t. That is,

$$\lim_{t \to \infty} y_{tr}(t) = 0.$$

Moreover, the steady state signal is the exponential,

$$\forall t, \quad y_{ss}(t) = H(\omega) e^{i\omega t} u(t). \tag{13.18}$$

Thus for stable systems, we can use the frequency response to describe the eventual output to sinusoidal signals that start at some finite time.

For the special case $\omega = 0$, the input (13.17) is the unit step, $x = u$, and $y_{ss} = H(0)u$. So for stable systems, the steady state response to a unit step input is a step of size $H(0)$. ($H(0)$ is called the **dc gain**.) This case is important in the design of feedback control, considered in the next chapter.

Let h be the impulse response and $\hat{H}$ be the Laplace transform of a stable and causal LTI system. We assume for simplicity that $\hat{H}$ is a strictly proper rational polynomial all of whose poles have multiplicity one,

$$\hat{H}(s) = \frac{A(s)}{(s-p_1)\cdots(s-p_N)} .$$

Because the system is causal, $RoC(h)$ has the form

$$RoC(h) = \{s \mid Re\{s\} > q\},$$

where q is the largest real part of any pole. Since the system is stable, $q < 0$, so that the region of convergence includes the imaginary axis.

From table 13.3 the Laplace transform $\hat{X}$ of the signal (13.17) is

$$\hat{X}(s) = \frac{1}{s - i\omega},$$

with $RoC(x) = \{s \in Complex \mid Re\{s\} > 0\}$.

The Laplace transform of the output $y = h * x$ is

$$\hat{Y} = \hat{H}\hat{X},$$

with

$$RoC(y) \supset RoC(h) \cap RoC(x) = \{s \in Complex \mid Re\{s\} > 0\}.$$

The partial fraction expansion of $\hat{Y}$ is

$$\begin{aligned} \hat{Y}(s) = \hat{H}(s)\hat{X}(s) &= \frac{A(s)}{(s-p_1)\cdots(s-p_N)} \cdot \frac{1}{s-i\omega} && (13.19) \\ &= \frac{R_1}{s-p_1} + \cdots + \frac{R_N}{s-p_N} + \frac{R_\omega}{s-i\omega} . && (13.20) \end{aligned}$$

Because everything is causal, each term must be causal, so from table 13.3 we obtain

$$\forall t, \quad y(t) = \sum_{k=1}^{N} R_k e^{p_k t} u(t) + R_\omega e^{i\omega t} u(t).$$

We decompose $y = y_{tr} + y_{ss}$, with

$$\begin{aligned} \forall t, \quad y_{tr}(t) &= \sum_{k=1}^{N} R_k e^{p_k t} u(t), \\ \forall t, \quad y_{ss}(t) &= R_\omega e^{i\omega t} u(t). \end{aligned}$$

Since $Re\{p_k\} < 0$ for $k = 1, \cdots, N$,

$$\lim_{t\to\infty} y_{tr}(t) = 0.$$

Thus, the steady-state response y_{ss} is eventually all that is left.

Finally, the residue R_ω is obtained by multiplying both sides of (13.19) by $s - i\omega$ and evaluating at $s = i\omega$ to get $R_\omega = \hat{H}(i\omega) = H(\omega)$, so that

$$\forall t, \quad y_{ss}(t) = H(\omega) e^{i\omega t} u(t).$$

This analysis reveals several interesting features of the total response y. First, from (13.20) we see the poles $p_1, \cdots, p_N$ of the transfer function contribute to the transient response y_{tr}, and the pole of the input $\hat{X}$ at $i\omega$ contributes to the steady state response. Second we can determine how quickly the transient response dies down. The transient response is

$$\forall t, \quad y_{tr}(t) = R_1 e^{p_1 t} u(t) + \cdots + R_N e^{p_N t} u(t).$$

The magnitude of the terms is

$$|R_1| e^{Re\{p_1\} t}, \cdots, |R_N| e^{Re\{p_N\} t}.$$

Each term decreases exponentially with t, since the real parts of the poles are negative. The slowest decrease is due to the pole with the least negative part. Thus the pole of the stable, causal transfer function with the least negative part determines how fast the transient response goes to zero. Indeed for large t, we can approximate the response y as

$$y(t) \approx R_i e^{p_i t} + H(\omega) e^{i\omega t},$$

where p_i is the pole with the largest (least negative) real part.

There is a similar result for discrete-time systems, and it is obtained in the same way. Suppose an exponential input

$$\forall n \in Integers, \quad x(n) = e^{i\omega n} u(n),$$

is applied to a stable and causal system with impulse response h, transfer function $\hat{H}$, and frequency response H. Then the output $y = h * x$ can again be decomposed as

$$\forall n, \quad y(n) = y_{tr}(n) + y_{ss}(n),$$

where the transient $y_{tr}(n) \to 0$ as $n \to \infty$, and the steady state response is

$$\forall n, \quad y_{ss}(n) = \hat{H}(e^{i\omega})e^{i\omega n}u(n) = H(\omega)e^{i\omega n}u(n).$$

For large n, the transient response decays exponentially as p_i^n, i.e.

$$y(n) \approx R_i p_i^n + y_{ss}(n),$$

where p_i is the pole with the largest magnitude (which must be less than one, since the system is stable).

13.7 Linear difference and differential equations

Many natural and man-made systems can be modeled as linear differential equations or difference equations. We have seen that when such systems are initially at rest, they are LTI systems. Hence, we can use their transfer functions (which are Z transforms or Laplace transforms) to analyze the response of these systems to external inputs.

However, physical systems are often not initially at rest. Dealing with non-zero initial conditions introduces some complexity in the analysis. Mathematicians call such systems with non-zero initial conditions **initial value problems**. We can adapt our methods to deal with initial conditions. The rest of this chapter is devoted to these methods.

Example 13.19: In example 13.8 we considered the LTI system described by the difference equation

$$y(n) - 0.9y(n-1) = x(n).$$

The transfer function of this system is $\hat{H}(z) = z/(z-0.9)$. If the system is initially at rest, we can calculate its response y from its Z transform $\hat{Y} = \hat{H}\hat{X}$. For instance, if the input is the unit step, $\hat{X}(z) = z/(z-1)$,

$$\hat{Y}(z) = \frac{z^2}{(z-0.9)(z-1)} = \frac{-9z}{z-0.9} + \frac{10z}{z-1},$$

and so $y(n) = -9(0.9)^n + 10, n \geq 0$.

We cannot use the transfer function, however, to determine the response if the initial condition at time $n = 0$ is $y(-1) = \overline{y}(-1)$, and the input is $x(n) = 0, n \geq 0$. The response to this initial condition is

$$y(n) = \overline{y}(-1)(0.9)^{n+1},\ n \geq -1.$$

We can check that this expression is correct by verifying that it satisfies both the initial condition and the difference equation.

If the initial condition is $y(-1) = \overline{y}(-1)$ and the input is a unit step, the response turns out to be the sum of the response due to the input (with zero initial condition) and the response due to the initial condition (with zero input),

$$y(n) = [-9(0.9)^n + 10] + [\overline{y}(-1)(0.9)^{n+1}],\ n \geq 0.$$

For small values of n the response depends heavily on the initial condition, especially if $\overline{y}(0)$ is large. Because this system is stable, the effect of the initial condition becomes vanishingly small for large n.

An **LTI difference equation** has the form

$$y(n)+a_1y(n-1)+\cdots+a_my(n-m)=b_0x(n)+\cdots+b_kx(n-k),\ n\geq 0. \tag{13.21}$$

We interpret this equation as describing a causal discrete-time LTI system in which $x(n)$ is the input and $y(n)$ is the output at time n. The a_i and b_j are constant coefficients that specify the system.

We have used difference equations before. In section 8.2.1 we used this form and the discrete time Fourier transform to find the frequency response of this system. In section 9.5 we showed how to realize such systems as IIR filters. In example 13.19 we used the transfer function to find the response. But in all these cases, we had to assume that the system was initially at rest. We now develop a method to find the response for arbitrary inital conditions.

We assume the input signal x starts at some finite time, which we take to be zero, $x(n)=0, n<0$. We wish to calculate $y(n), n\geq 0$. From (13.21) we can see that we need to be given m initial conditions,

$$y(-1)=\overline{y}(-1),\cdots,y(-m)=\overline{y}(-m).$$

Given the input signal and these initial conditions, there is a straightforward procedure to calculate the output response $y(n), n\geq 0$: Rewrite (13.21) as

$$y(n)=-a_1y(n-1)-\cdots-a_my(n-m)+b_0x(n)+\cdots+b_kx(n-k), \tag{13.22}$$

and recursively use (13.22) to obtain $y(0),y(1),y(2),\cdots$. For $n=0$, (13.22) yields

$$\begin{aligned} y(0) &= -a_1y(-1)-\cdots-a_my(-m)+b_0x(0)+\cdots+b_kx(-k) \\ &= -a_1\overline{y}(-1)-\cdots-a_m\overline{y}(-m)+b_0x(0). \end{aligned}$$

All the terms on the right are known from the initial conditions and the input $x(0)$, so we can calculate $y(0)$. Next, taking $n=1$ in (13.22),

$$y(1)=-a_1y(0)+\cdots+a_my(1-m)+b_0x(1)+\cdots+b_kx(1-k).$$

All the terms on the right are known either from the given data or from precalculated values—$y(0)$ in this case. Proceeding in this way we can calculate the remaining values of the output sequence $y(2),y(3),\cdots$, one at a time.

We now use the Z transform to calculate the *entire* output sequence. Multiplying both sides of (13.21) by $u(n)$, the unit step, gives us a relation that holds among signals whose domain is *Integers*:

$$y(n)u(n)+a_1y(n-1)u(n)+\cdots+a_my(n-m)u(n)=b_0x(n)u(n)+\cdots+b_kx(n-k)u(n),\ n\in Integers.$$

We can now take the Z transforms of both sides. We multiply both sides by z^{-n} and sum,

$$\sum_{n=0}^{\infty}y(n)z^{-n}+a_1\sum_{n=0}^{\infty}y(n-1)z^{-n}+\cdots+a_m\sum_{n=0}^{\infty}y(n-m)z^{-n}=b_0\sum_{n=0}^{\infty}x(n)z^{-n}+\cdots+b_k\sum_{n=0}^{\infty}x(n-k)z^{-n}. \tag{13.23}$$

Define

$$\hat{X}(z) = \sum_{n=0}^{\infty} x(n)z^{-n}, \quad \hat{Y}(z) = \sum_{n=0}^{\infty} y(n)z^{-n}.$$

Each sum in (13.23) can be expressed in terms of $\hat{Y}$ or $\hat{X}$. In evaluting the Z transforms of the signals $y(n-1)u(n), y(n-2)u(n), \cdots$ we need to include the initial conditions:

$$\begin{aligned}
\sum_{n=0}^{\infty} y(n-1)z^{-n} &= \overline{y}(-1)z^0 + z^{-1}\sum_{n=1}^{\infty} y(n-1)z^{-(n-1)} = \overline{y}(-1)z^0 + z^{-1}\hat{Y}(z),\\
\sum_{n=0}^{\infty} y(n-2)z^{-n} &= \overline{y}(-2)z^0 + \overline{y}(-1)z^{-1} + z^{-2}\sum_{n=2}^{\infty} y(n-2)z^{-(n-2)}\\
&= \overline{y}(-2)z^0 + \overline{y}(-1)z^{-1} + z^{-2}\hat{Y}(z),\\
&\cdots\\
\sum_{n=0}^{\infty} y(n-m)z^{-n} &= \overline{y}(-m)z^0 + \cdots + \overline{y}(-1)z^{-(m-1)} + z^{-m}\sum_{n=m}^{\infty} y(n-m)z^{-(n-m)}\\
&= \overline{y}(-m)z^0 + \cdots + \overline{y}(-1)z^{-(m-1)} + z^{-m}\hat{Y}(z).
\end{aligned}$$

Because $x(n) = 0, n < 0$, by assumption, the sums on the right in (13.23) are simpler:

$$\begin{aligned}
\sum_{n=0}^{\infty} x(n-1)z^{-n} &= x(-1)z^0 + z^{-1}\hat{X}(z) = z^{-1}\hat{X}(z)\\
\sum_{n=0}^{\infty} x(n-2)z^{-n} &= x(-2)z^0 + x(-1)z^{-1} + z^{-2}\hat{X}(z) = z^{-2}\hat{X}(z)\\
&\cdots\\
\sum_{n=0}^{\infty} x(n-k)z^{-n} &= x(-k)z^0 + \cdots + x(-1)z^{-(k-1)} + z^{-k}\hat{X}(z) = z^{-k}\hat{X}(z).
\end{aligned}$$

(If there were non-zero initial conditions for $x(-1), \cdots, x(-k)$, we could include them in the Z transforms of $x(n-1)u(n), \cdots, x(n-k)u(n)$.) Substituting these relations in (13.23) yields

$$\begin{aligned}
\hat{Y}(z) \quad &+ \quad a_1[z^{-1}\hat{Y}(z) + \overline{y}(-1)z^0] + \cdots + a_m[z^{-m}\hat{Y}(z) + \overline{y}(-m)z^0 + \cdots \overline{y}(-1)z^{-(m-1)}]\\
&= \quad b_0\hat{X}(z) + b_1 z^{-1}\hat{X}(z) + \cdots b_k\hat{X}z^{-k},
\end{aligned} \tag{13.24}$$

from which, by rearranging terms, we obtain

$$[1 + a_1 z^{-1} + \cdots + a_m z^{-m}]\hat{Y}(z) = [b_0 + b_1 z^{-1} + \cdots + b_k z^{-k}]\hat{X}(z) + \hat{C}(z),$$

where $\hat{C}(z)$ is an expression involving only the initial conditions $\overline{y}(-1), \cdots, \overline{y}(-m)$. Therefore,

$$\hat{Y}(z) = \frac{b_0 + b_1 z^{-1} + \cdots b_k z^{-k}}{1 + a_1 z^{-1} + \cdots + a_m z^{-m}}\hat{X}(z) + \frac{\hat{C}(z)}{1 + a_1 z^{-1} + \cdots + a_m z^{-m}}.$$

We rewrite this relation as

$$\hat{Y}(z) = \hat{H}(z)\hat{X}(z) + \frac{\hat{C}(z)}{1 + a_1 z^{-1} + \cdots + a_m z^{-m}}. \tag{13.25}$$

where

$$\boxed{\hat{H}(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_k z^{-k}}{1 + a_1 z^{-1} + \cdots + a_m z^{-m}}}\,. \tag{13.26}$$

Observe that if the initial conditions are all zero, $\hat{C}(z) = 0$, and we only have the first term on the right in (13.25); and if the input is zero—i.e., $x(n) = 0$ for all n, then $\hat{X}(z) = 0$, and we only have the second term.

By definition, $\hat{Y}(z)$ is the Z transform of the causal signal $y(n)u(n), n \in Integers$. So its $RoC = \{z \in Complex \mid |z| > |p|\}$ in which p is the pole of the right side of (13.25) with the largest magnitude. The inverse Z transform of $\hat{Y}$ can be expressed as

$$\forall n \geq 0, \quad y(n) = y_{zs}(n) + y_{zi}(n), \tag{13.27}$$

where $y_{zs}(n)$, the inverse Z transform of $\hat{H}\hat{X}$, is the **zero-state response**, and $y_{zi}(n)$, the inverse Z transform of $\hat{C}(z)/[1 + a_1 z^{-1} + \cdots + a_m z^{-m}]$, is the **zero-input response**. The zero-state response, also called the **forced response**, is the output when all initial conditions are zero. The zero-input response, also called the **natural response**, is the output when the input is zero.

Thus the (total) response is the sum of the zero-state and zero-input response. We first encountered this property of linearity in chapter 5.

By definition, the **transfer function** is the Z transform of the zero-state impulse response. Taking $\hat{C} = 0$ and $\hat{X} = 1$ in (13.25) shows that the transfer function is $\hat{H}(z)$. From (13.26) we see that $\hat{H}$ can be written down by inspection of the difference equation (13.21). If the system is stable—all poles of $\hat{H}$ are inside the unit circle—the frequency response is

$$\forall \omega, \quad H(\omega) = \hat{H}(e^{i\omega}) = \frac{b_0 + b_1 e^{-i\omega} + \cdots + b_k e^{-ik\omega}}{1 + a_1 e^{-i\omega} + \cdots + a_m e^{-im\omega}}.$$

We saw this relation in (8.21).

Example 13.20: Consider the difference equation

$$y(n) - \frac{5}{6}y(n-1) + \frac{1}{6}y(n-2) = x(n), \; n \geq 0.$$

Taking Z transforms as in (13.24) yields

$$\hat{Y}(z) - \frac{5}{6}[z^{-1}\hat{Y}(z) + \overline{y}(-1)] + \frac{1}{6}[z^{-2}\hat{Y}(z) + \overline{y}(-2) + \overline{y}(-1)z^{-1}] = \hat{X}(z).$$

Therefore

$$\begin{aligned}
\hat{Y}(z) &= \frac{1}{1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}}\hat{X}(z) + \frac{\frac{5}{6}\overline{y}(-1) + \frac{1}{6}\overline{y}(-2) + \frac{1}{6}\overline{y}(-1)z^{-1}}{1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}} \\
&= \frac{z^2}{z^2 - \frac{5}{6}z + \frac{1}{6}}\hat{X}(z) + \frac{[\frac{5}{6}\overline{y}(-1) + \frac{1}{6}\overline{y}(-2)]z^2 + \frac{1}{6}\overline{y}(-1)z}{z^2 - \frac{5}{6}z + \frac{1}{6}}\,,
\end{aligned}$$

from which we can obtain $\hat{Y}$ for a specified $\hat{X}$ and initial conditions $\overline{y}(-1), \overline{y}(-2)$. The transfer function is

$$\hat{H}(z) = \frac{z^2}{z^2 - \frac{5}{6}z + \frac{1}{6}} = \frac{z^2}{(z - \frac{1}{3})(z - \frac{1}{2})},$$

which has poles at $z = 1/3$ and $z = 1/2$ (and two zeros at $z = 0$). The system is stable. The zero-state impulse response h is the inverse Z transform of $\hat{H}(z)$, which we obtain using partial fraction expansion,

$$\hat{H}(z) = z\left[\frac{-2}{z-\frac{1}{3}} + \frac{3}{z-\frac{1}{2}}\right]$$

so that

$$\forall n \in Integers, \quad h(n) = -2\left(\frac{1}{3}\right)^n u(n) + 3\left(\frac{1}{2}\right)^n u(n).$$

We can recognize that the impulse response consists of two terms, each contributed by one pole of the transfer function.

Suppose the initial conditions are $\overline{y}(-1) = 1, \overline{y}(-2) = 1$ and the input x is the unit step, so $\hat{X}(z) = z/(z-1)$. Then the zero-input response, y_{zi}, has Z transform

$$\begin{aligned}\hat{Y}_{zi}(z) &= \frac{[\frac{5}{6}\overline{y}(-1) + \frac{1}{6}\overline{y}(-2)]z^2 + \frac{1}{6}\overline{y}(-1)z}{(z-\frac{1}{3})(z-\frac{1}{2})} \\ &= \frac{z^2 + \frac{1}{6}z}{(z-\frac{1}{3})(z-\frac{1}{2})} = z\left[\frac{-3}{z-\frac{1}{3}} + \frac{4}{z-\frac{1}{2}}\right],\end{aligned}$$

so

$$\forall n, \quad y_{zi}(n) = -3\left(\frac{1}{3}\right)^n u(n) + 4\left(\frac{1}{2}\right)^n u(n).$$

The zero-state response, y_{zs}, has Z transform

$$\begin{aligned}\hat{Y}_{zs}(z) &= \hat{H}(z)\hat{X}(z) = \frac{z^3}{(z-\frac{1}{3})(z-\frac{1}{2})(z-1)} \\ &= z\left[\frac{1}{z-\frac{1}{3}} + \frac{-3}{z-\frac{1}{2}} + \frac{3}{z-1}\right],\end{aligned}$$

so

$$\forall n, \quad y_{zs}(n) = \left(\frac{1}{3}\right)^n u(n) - 3\left(\frac{1}{2}\right)^n u(n) + 3u(n).$$

The (total) response

$$\forall n \in Integers, \quad y(n) = y_{zs}(n) + y_{zi}(n) = 3u(n) + [-2(1/3)^n + (1/2)^n]u(n),$$

can also be expressed as the sum of the steady-state and the transient response with $y_{ss}(n) = 3u(n)$ and $y_{tr}(n) = -2(1/3)^n u(n) + (1/2)^n u(n)$. Note that the decomposition of the response into the sum of the zero-state and zero-input responses is different from its decomposition into the steady-state and transient responses.

13.7.1 LTI differential equations

The analogous development for continuous time concerns systems described by a **LTI differential equation** of the form

$$\frac{d^m y}{dt^m}(t) + a_{m-1}\frac{d^{m-1}y}{dt^{m-1}}(t) + \cdots + a_1\frac{dy}{dt}(t) + a_0 y(t) = b_k\frac{d^k x}{dt^k}(t) + \cdots + b_1\frac{dx}{dt}(t) + b_0 x(t),\ t \geq 0. \tag{13.28}$$

We interpret this equation as describing a causal continuous-time LTI system in which $x(t)$ is the input and $y(t)$ is the output at time t. The constant coefficients a_i and b_j specify the system.

In section 8.2.1 we used this form to find the frequency response. In example 13.11, we used the Laplace transform to find the transfer function of a tuning force. But in both cases, we assumed that the system was initially at rest. We now develop a method to find the response to arbitrary initial conditions. We begin with a simple circuit example.

Example 13.21: A series connection of a resistor R, a capacitor C, and a voltage source x, is described by the differential equation

$$\frac{dy}{dt}(t) + \frac{1}{RC}y(t) = x(t),$$

in which y is the voltage across the capacitor. The differential equation is obtained from Kirchhoff's voltage law. The transfer function of this system is $\hat{H}(s) = 1/(s+1/RC)$. So if the system is initially at rest, we can calculate the response y from its Laplace transform $\hat{Y} = \hat{H}\hat{X}$. For instance, if the input is a unit step, $\hat{X}(s) = 1/s$,

$$\hat{Y}(s) = \frac{1}{(s+1/RC)s} = \frac{-RC}{s+1/RC} + \frac{RC}{s},$$

therefore, $y(t) = -RCe^{-t/RC} + RC,\ t \geq 0$.

We cannot use this transfer function, however, to determine the response if the initial capacitor voltage is $y(0) = \bar{y}(0)$ and $x(t) = 0, t \geq 0$. The response in this case is

$$y(t) = \bar{y}(0)e^{-t/RC},\ t \geq 0.$$

We can check that expression is correct by verifying that it satisfies the given initial condition and the differential equation.

If the initial condition is $y(0) = \bar{y}(0)$ and the input is a unit step, the response turns out to be the sum of the response due to the input (with zero initial condition) and the response due to the initial condition (with zero input),

$$y(t) = [-RCe^{-t/RC} + RC] + [\bar{y}(0)e^{-t/RC}],\ t \geq 0.$$

For the general case (13.28) we assume that the input x starts at some finite time which we take to be zero, so $x(t) = 0, t < 0$. We wish to calculate $y(t), t \geq 0$. From the theory of differential equations we know that we need to be given m initial conditions,

$$y(0) = \bar{y}(0), \frac{dy}{dt}(0) = \bar{y}^{(1)}(0), \cdots, \frac{d^{m-1}y}{dt^{m-1}}(0) = \bar{y}^{(m-1)}(0),$$

in order to calculate $y(t), t \geq 0$.

Because time is continuous, there is no recursive procedure for calculating the output from the given data as we did in (13.22). Instead we calculate the output signal using the Laplace transform. We define the Laplace transforms of the signals $y(t)u(t), y^{(1)}(t)u(t), \cdots, y^{(m)}(t)u(t), x(t)u(t)$:

$$\begin{aligned}
\hat{Y}(s) &= \int_{-\infty}^{\infty} y(t)u(t)e^{-st}dt = \int_{0}^{\infty} y(t)e^{-st}dt \\
\hat{Y}^{(i)}(s) &= \int_{-\infty}^{\infty} y^{(i)}(t)u(t) = \int_{0}^{\infty} y^{(i)}(t)e^{-st}dt, i = 1, \cdots, m \\
\hat{X}(s) &= \int_{-\infty}^{\infty} x(t)u(t)e^{-st}dt = \int_{0}^{\infty} x(t)e^{-st}dt.
\end{aligned}$$

Here we use the notation $y^{(i)}(t) = \frac{d^i t}{dt^i}y(t), t \geq 0$. We now derive the relations between these Laplace transforms.

The derivative $y^{(1)}(t) = \frac{dy}{dt}(t)$ and y are related by

$$y(t)u(t) = y(0)u(t) + \int_{0}^{t} y^{(1)}(\tau)u(\tau)d\tau = \overline{y}(0)u(t) + \int_{0}^{t} y^{(1)}(\tau)u(\tau)d\tau, \ t \in Reals.$$

Using integration by parts,

$$\begin{aligned}
\hat{Y}(s) &= \int_{0}^{\infty} y(t)e^{-st}dt = \int_{0}^{\infty} \overline{y}(0)e^{-st}dt + \int_{0}^{\infty} \left(\int_{0}^{t} y^{(1)}(\tau)d\tau\right) e^{-st}dt \\
&= \frac{1}{s}\overline{y}(0) - \frac{1}{s}\int_{0}^{t} y^{(1)}(\tau)d\tau e^{-st} \,|_{t=0}^{\infty} + \frac{1}{s}\int_{0}^{\infty} y^{(1)}(t)e^{-st}dt \\
&= \frac{1}{s}[\hat{Y}^{(1)}(s) + \overline{y}(0)].
\end{aligned}$$

Therefore,

$$\boxed{\hat{Y}^{(1)}(s) = s\hat{Y}(s) - \overline{y}(0).} \qquad (13.29)$$

Repeating this procedure, we get the Laplace transforms of the higher-order derivatives,

$$\begin{aligned}
\hat{Y}^{(2)}(s) &= s\hat{Y}^{(1)}(s) - \overline{y}^{(1)}(0) \\
&= s^2\hat{Y}(s) - s\overline{y}(0) - \overline{y}^{(1)}(0) \\
&\cdots \\
\hat{Y}^{(m)}(s) &= s^m\hat{Y}(s) - s^{m-1}\overline{y}(0) - s^{m-2}\overline{y}^{(1)}(0) - \cdots - \overline{y}^{(m-1)}(0).
\end{aligned}$$

On the other hand, because $x^{(i)}(t) = \frac{d^i x}{dt^i}(t)$ for all $t \in Reals$, using the differentiation property in table 13.4, we obtain

$$\begin{aligned}
\hat{X}^{(1)}(s) &= s\hat{X}(s) \\
&\cdots \\
\hat{X}^{(k)}(s) &= s^k\hat{X}(s).
\end{aligned}$$

By substituting from the relations just derived, we obtain the Laplace transforms of all the terms in (13.28),

$$[s^m\hat{Y}(s) - s^{m-1}\overline{y}(0) - \cdots - \overline{y}^{m-1}(0)] + a_{m-1}[s^{m-1}\hat{Y}(s) - s^{m-2}\overline{y}(0) - \cdots - \overline{y}^{m-2}(0)]$$
$$\cdots + a_1[s\hat{Y}(s) - \overline{y}(0)] + a_0\hat{Y}(s) = b_k s^k \hat{X}(s) + \cdots + b_1 s\hat{X}(s) + b_0\hat{X}(s). \tag{13.30}$$

Rearranging terms yields

$$[s^m + a_{m-1}s^{m-1} + \cdots + a_1 s + a_0]\hat{Y}(s) = [b_k s^k + \cdots + b_1 s + b_0]\hat{X}(s) + \hat{C}(s),$$

in which $\hat{C}$ is an expression involving only the intial conditions $\overline{y}(0), \cdots, \overline{y}^{(m-1)}(0)$. Therefore,

$$\hat{Y}(s) = \frac{b_k s^k + b_{k-1}s^{k-1} + \cdots b_1 s + b_0}{s^m + a_{m-1}s^{m-1} + \cdots a_1 s + a_0}\, \hat{X}(s) + \frac{\hat{C}(s)}{s^m + a_{m-1}s^{m-1} + \cdots a_1 s + a_0}, \tag{13.31}$$

which we also write as

$$\hat{Y}(s) = \hat{H}(s)\hat{X}(s) + \frac{\hat{C}(s)}{s^m + a_{m-1}s^{m-1} + \cdots a_1 s + a_0}, \tag{13.32}$$

in which

$$\boxed{\hat{H}(s) = \frac{b_k s^k + \cdots + b_1 s + b_0}{s^m + \cdots + a_1 s + a_0}.} \tag{13.33}$$

If the initial conditions are all zero, $\hat{C}(s) = 0$, and we only have the first term on the right in (13.32); if the input is zero—i.e., $x(t) = 0$ for all t, then $\hat{X}(s) = 0$, and we only get the second term in (13.32).

By definition, $\hat{Y}(s)$ is the Laplace transform of the causal signal $y(t)u(t), t \in Reals$. So its $RoC = \{s \in Complex \mid Re\{s\} > Re\{p\}\}$, where p is a pole of the right side of (13.32) with the largest real part.

Taking the inverse Laplace transform of $\hat{Y}$, we can decompose the output signal y as

$$\forall t, \quad y(t) = y_{zs}(t) + y_{zi}(t),$$

where y_{zs}, the inverse Laplace transform of $\hat{H}\hat{X}$, is the **zero-state** or **forced response** and y_{zi}, the inverse Laplace transform of $\hat{C}(s)/[s^m + \cdots + a_0]$, is the **zero-input** or **natural response**. The (total) response is the sum of the zero-state and zero-input response, which is a general property of linear systems.

By definition, the **transfer function** is the Laplace transform of the zero-state impulse response. Taking $\hat{C} = 0$ and $\hat{X} = 1$ (the Laplace transform of the unit impulse) in (13.32) shows that the transfer function is $\hat{H}(s)$ which, as we see from (13.33), can be written down by inspection of the differential equation (13.28). If the system is stable—all poles of $\hat{H}(s)$ have real parts strictly less than zero—the frequency response is

$$\forall \omega, \quad H(\omega) = \hat{H}(i\omega) = \frac{b_k(i\omega)^k + \cdots + b_1 i\omega + b_0}{(i\omega)^m + \cdots + a_1 i\omega + a_0}.$$

We saw this relation in (**??**).

Example 13.22: We find the response $y(t), t \geq 0$, for the differential equation

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 3x(t) + \frac{dx}{dt},$$

when the input is a unit step $x(t) = u(t)$ and the initial conditions are $y(0) = 1, y^{(1)}(0) = 2$. Taking Laplace transforms of both sides as in (13.30),

$$[s^2\hat{Y}(s) - s\overline{y}(0) - \overline{y}^{(1)}(0)] + 3[s\hat{Y}(s) - \overline{y}(0)] + 2\hat{Y}(s) = 3\hat{X}(s) + s\hat{X}(s).$$

Therefore,

$$\hat{Y}(s) = \frac{s+3}{s^2+3s+2}\hat{X}(s) + \frac{s\overline{y}(0) + \overline{y}^{(1)}(0) + 3\overline{y}(0)}{s^2+3s+2}.$$

Substituting $\hat{X}(s) = 1/s, \overline{y}(0) = 1, \overline{y}^{(1)} = 2$, yields

$$\begin{aligned}
\hat{Y}(s) &= \frac{s+3}{s(s^2+3s+2)} + \frac{s+5}{s^2+3s+2} \\
&= [\frac{3/2}{s} - \frac{2}{s+1} + \frac{1/2}{s+2}] + [\frac{4}{s+1} - \frac{3}{s+2}].
\end{aligned}$$

Taking inverse Laplace transforms gives

$$\begin{aligned}
\forall t, \quad y(t) &= y_{zs}(t) + y_{zi}(t) \\
&= [\frac{3}{2}u(t) - 2e^{-t}u(t) + \frac{1}{2}e^{-2t}u(t)] + [4e^{-t}u(t) - 3e^{-2t}u(t)] \\
&= \frac{3}{2}u(t) + [2e^{-t} - \frac{5}{2}e^{-2t}]u(t) \\
&= y_{ss}(t) + y_{tr}(t).
\end{aligned}$$

As in the case of difference equations, the decomposition of the response into zero-state and zero-input responses is different from the decomposition into transient and steady-state responses. (Indeed, the steady-state response does not exist if the system is unstable, whereas the former decomposition always exists.)

13.8 State-space models

This section is mathematically more advanced in that it uses the operation of matrix inverse.

In section 5.3 we introduced single-input, single-output (SISO) multidimensional state-space models of discrete-time and continuous-time LTI systems. For LTI systems, state-space models provide an alternative description to difference or differential equation representations. The advantage of state-space models is that by using matrix notation we have a very compact representation of the response, independent of the order of the system. We develop a method that combines this matrix notation with transform techniques to calculate the response.

The discrete-time SISO state-space model is

$$\begin{aligned}
\forall\, n \geq 0, \quad s(n+1) &= As(n) + bx(n), && (13.34) \\
y(n) &= c^T s(n) + dx(n), && (13.35)
\end{aligned}$$

in which $s(n) \in Reals^N$ is the state, $x(n) \in Reals$ is the input, and $y(n) \in Reals$ is the output at time n. In this $[A,b,c,d]$ representation, A is an $N \times N$ (square) matrix, b,c are N-dimensional column vectors, and d is a scalar. If the initial state is $s(0)$, and the input sequence is $x(0), x(1), \cdots$, by recursively using (13.34) and (13.35) we obtain the state and output responses:

$$s(n) = A^n s(0) + \sum_{m=0}^{n-1} A^{n-1-m} b x(m), \tag{13.36}$$

$$y(n) = c^T A^n s(0) + \{\sum_{m=0}^{n-1} c^T A^{n-1-m} b x(m) + d x(n)\}, \tag{13.37}$$

for all $n \geq 0$. Notice that these "closed-form" formulas for the response are independent of the order N. Difference equation representations do not have such a closed-form formula.

Example 13.23: Consider the system described by the difference equation

$$y(n) - 2y(n-1) - 3y(n-2) = x(n).$$

As in section 5.3, we can construct a state-space model for this system by noting that the state at time n should remember the previous two inputs $y(n-1), y(n-2)$. Define the two-dimensional state vector $s(n) = [s_1(n) \ s_2(n)]^T$ by $s_1(n) = y(n-1), s_2(n) = ay(n-2)$, in which $a \neq 0$ is a constant. Problem 23 at the end of this chapter asks you to show that the $[A,b,c,d]$ representation for this choice of state is given by

$$A = \begin{bmatrix} 2 & 3/a \\ a & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad c^T = [2 \quad 3/a], \text{ and } d = 1.$$

Different choices of a give a different state-space model. However, they all have the same input-output relation because they all have the same transfer function.

We will obtain the Z transforms of the response sequences (13.36), (13.37). The key is to compute the Z transform of the entire $N \times N$ matrix sequence $A^n u(n), n \in Integers$. This Z transform is

$$\boxed{\sum_{n=0}^{\infty} z^{-n} A^n = [I - z^{-1}A]^{-1} = z[zI - A]^{-1}.} \tag{13.38}$$

Here z is a complex number and I is the $N \times N$ identity matrix. The series on the left is an infinite sum of $N \times N$ matrices which converges to the $N \times N$ matrix on the right, for $z \in RoC$. RoC is determined later.

Assuming the series converges, it is easy to check the equality (13.38): Just multiply both sides by $[I - z^{-1}A]$ and verify that

$$[I - z^{-1}A] \sum_{n=0}^{\infty} z^{-n} A^n = \sum_{n=0}^{\infty} z^{-n} A^{-n} - \sum_{n=0}^{\infty} z^{-(n+1)} A^{n+1} = z^0 A^0 = I.$$

Next, denote by F the matrix inverse,

$$F(z) = [I - z^{-1}A]^{-1} = z[zI - A]^{-1}, \tag{13.39}$$

and the coefficients of A^n and $F(z)$ by

$$A^n = [a_{ij}(n) \mid 1 \le i, j \le N], \quad F(z) = [f_{ij}(z) \mid 1 \le i, j \le N].$$

Then $f_{ij}(z) = \sum_{n=0}^{\infty} z^{-n} a_{ij}(n)$ is the Z transform of the sequence $a_{ij}(n), n \ge 0$, $1 \le i, j \le N$. So we can obtain $A^n, n \ge 0$, by taking the inverse Z transform of each of the N^2 coefficients of $F(z)$.

Example 13.24: Let

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix},$$

then

$$[zI - A]^{-1} = \begin{bmatrix} z-2 & -1 \\ -3 & z-4 \end{bmatrix}^{-1} = \frac{1}{\det[zI-A]} \begin{bmatrix} z-4 & 1 \\ 3 & z-2 \end{bmatrix},$$

in which $\det[zI - A]$ denotes the determinant of $[zI - A]$,

$$\det[zI - A] = (z-2)(z-4) - 3 = z^2 - 6z + 5 = (z-1)(z-5).$$

Hence,

$$F(z) = z[zI - A]^{-1} = \frac{z}{(z-1)(z-5)} \begin{bmatrix} z-4 & 1 \\ 3 & z-2 \end{bmatrix} = \begin{bmatrix} \frac{z(z-4)}{(z-1)(z-5)} & \frac{z}{(z-1)(z-5)} \\ \frac{3z}{(z-1)(z-5)} & \frac{z(z-2)}{(z-1)(z-5)} \end{bmatrix}.$$

The partial fraction expansion of the coefficients of F is

$$F(z) = \begin{bmatrix} \frac{(3/4)z}{z-1} + \frac{(1/4)z}{z-5} & \frac{(-1/4)z}{z-1} + \frac{(1/4)z}{z-5} \\ \frac{(-3/4)z}{z-1} + \frac{(3/4)z}{z-5} & \frac{(1/4)z}{z-1} + \frac{(3/4)z}{z-5} \end{bmatrix}.$$

Using table 13.1 we find the inverse Z transform of every coefficient of $F(z)$: for all $n \in Integers$,

$$A^n u(n) \quad = \quad \begin{bmatrix} \frac{3}{4}u(n) + \frac{1}{4}5^n u(n) & -\frac{1}{4}u(n) + \frac{1}{4}5^n u(n) \\ -\frac{3}{4}u(n) + \frac{3}{4}5^n u(n) & \frac{1}{4}u(n) + \frac{3}{4}5^n u(n) \end{bmatrix}.$$

This is more revealingly expressed as

$$A^n = \begin{bmatrix} 3/4 & -1/4 \\ -3/4 & 1/4 \end{bmatrix} + 5^n \begin{bmatrix} 1/4 & 1/4 \\ 3/4 & 3/4 \end{bmatrix}, \quad n \ge 0,$$

because it shows that the variation in n of A^n is determined by the two poles, at $z = 1$ and $z = 5$, in the coefficients of $F(z)$. Moreover, these two poles are the zeros of

$$\det[zI - A] = (z-1)(z-5).$$

This determinant is called the **characterstic polynomial** of the matrix A and its zeros are called the **eigenvalues** of A. The domain of convergence is $RoC = \{z \in Complex \mid |z| > 5\}$.

We return to the general case in (13.39). Denote the matrix inverse of $[zI-A]$ as

$$[zI-A]^{-1} = \frac{1}{\det[zI-A]}G(z),$$

in which $G(z)$ is the $N \times N$ matrix of co-factors of $[zI-A]$. It follows that each coefficient $f_{ij}(z)$ of $F(z) = z[zI-A]^{-1}$ is a rational polynomial whose denominator is the characteristic polynomial of A, $\det[zI-A]$. Therefore, if there are no pole-zero cancellations, all coefficients of $F(z)$ have the same poles, which are the zeros of $\det[zI-A]$. These zeros are called the eigenvalues of A. The polynomial $\det[zI-A]$ is of order N, and so A has N eigenvalues.

Because $A^n u(n), n \in Integers$, is a causal sequence, the region of convergence is $RoC = \{z \in Complex \mid |z| > |p|\}$, in which p is the pole of F (or eigenvalue of A) with the largest magnitude. For the system (13.34), (13.35) to be stable, the poles of F must have magnitudes strictly smaller than 1.

Suppose A has N distinct eigenvalues $p_1, \cdots, p_N$,

$$\det[zI-A] = (z-p_1)\cdots(z-p_N).$$

Then the partial fraction expansion of $F(z)$ has the form

$$F(z) = \frac{z}{z-p_1}R_1 + \cdots + \frac{z}{z-p_N}R_N,$$

in which R_i is the matrix of residues of the coefficients of F at the pole p_i. R_i is a constant matrix, possibly with complex coefficients if p_i is complex. Recalling that $\frac{z}{z-p_i}$ is the inverse Z transform of $p_i^n u(n)$, we can take the inverse Z transform of $F(z)$ to conclude that

$$\boxed{A^n = p_1^n R_1 + \cdots p_N^n R_N, \quad n \geq 0.} \tag{13.40}$$

Thus A^n is a linear combination of $p_1^n, \cdots, p_N^n$.

We can decompose the response (13.37) into the zero-input and zero-state responses, expressing the latter as a convolution sum,

$$y(n) = c^T A^n s(0) + \sum_{m=0}^{n} h(n-m)x(m),\ n \geq 0,$$

where the (zero-state) impulse response is

$$h(n) = \begin{cases} 0, & n < 0 \\ d, & n = 0 \\ c^T A^{n-1} b, & n \geq 1 \end{cases}.$$

Let $\hat{X}, \hat{Y}, \hat{H}, \hat{Y}_{zi}$ be the Z transforms:

$$\hat{X}(z) = \sum_{n=0}^{\infty} x(n)z^{-n}, \quad \hat{Y}(z) = \sum_{n=0}^{\infty} y(n)z^{-n}, \quad \hat{H}(z) = \sum_{n=0}^{\infty} h(n)z^{-n}, \quad \hat{Y}_{zi}(z) = \sum_{n=0}^{\infty} c^T z^{-n} A^n s(0).$$

Then

$$\hat{Y} = \hat{H}\hat{X} + \hat{Y}_{zi}.$$

Because $\sum_{n=0}^{\infty} z^{-n} A^n = z[zI - A]^{-1}$, we obtain

$$\boxed{\hat{H}(z) = c^T [zI - A]^{-1} b + d,}$$

and

$$\boxed{\hat{Y}_{zi}(z) = z c^T [zI - A]^{-1} s(0).}$$

Example 13.25: Suppose A is as in example 13.24, $b^T = [1 \;\; 1], c^T = [2 \;\; 0], d = 3$, and $(s(0))^T = [0 \;\; 4]$. Then the transfer function is

$$\hat{H}(z) = [2 \;\; 0] \begin{bmatrix} \frac{(z-4)}{(z-1)(z-5)} & \frac{1}{(z-1)(z-5)} \\ \frac{3}{(z-1)(z-5)} & \frac{(z-2)}{(z-1)(z-5)} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 3 = \frac{2(z-4)+2}{(z-1)(z-5)} + 3,$$

and the Z transform of the zero-input response is

$$\hat{Y}_{zi}(z) = [2 \;\; 0] \begin{bmatrix} \frac{z(z-4)}{(z-1)(z-5)} & \frac{z}{(z-1)(z-5)} \\ \frac{3z}{(z-1)(z-5)} & \frac{z(z-2)}{(z-1)(z-5)} \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \frac{8z}{(z-1)(z-5)}.$$

The transfer function

$$\hat{H}(z) = \frac{2(z-4)+2}{(z-1)(z-5)} + 3 = \frac{3z^2 - 16z + 9}{z^2 - 6z + 5} = \frac{3 - 16z^{-1} + 9z^{-2}}{1 - 6z^{-1} + 5z^{-2}}.$$

From (13.26) we recognize that $\hat{H}$ is also the transfer function of the difference equation

$$y(n) - 6y(n-1) + 5y(n-2) = 3x(n) - 16x(n-1) + 9x(n-2).$$

This difference equation describes the same input-output relation as the state-space model of this example.

13.8.1 Continuous-time state-space models

The continuous-time SISO state-space model introduced in section 5.4 has the $[A, b, c, d]$ representation

$$\dot{v}(t) = Av(t) + bx(t), \tag{13.41}$$

$$y(t) = c^T v(t) + dx(t), \tag{13.42}$$

in which $v(t) \in Reals^N$ is the state, $x(t) \in Reals$ is the input, and $y(t) \in Reals$ is the output at time $t \in Reals$. A is an $N \times N$ matrix, and b, c are N-dimensional column vectors, and d is a scalar. (We use v instead of s to denote the state, because s is reserved for the Laplace transform variable.)

Given the initial state $v(0)$ and the input signal $x(t), t \geq 0$, we will show that the state response and the output response obey the formulas

$$v(t) = e^{tA}v(0) + \int_0^t e^{(t-\tau)A}bx(\tau)d\tau, \tag{13.43}$$

$$y(t) = c^T e^{tA}v(0) + [\int_0^t c^T e^{(t-\tau)A}bx(\tau)d\tau] + dx(t). \tag{13.44}$$

In these formulas, e^{tA} or $\exp(tA)$ is the name of the $N \times N$ matrix

$$e^{tA} = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} = I + tA + \frac{(tA)^2}{2!} + \frac{(tA)^3}{3!} + \cdots, \tag{13.45}$$

where $(tA)^k$ is the matrix tA multiplied by itself k times, and $(tA)^0 = I$, the $N \times N$ identity matrix. Definition (13.45) of the **matrix exponential** is the natural generalization of the exponential of a real or complex number. (The series in (13.45) is absolutely summable because of the factor $k!$ in the denominator.)

Unlike in the discrete-time case, there is no recursive procedure to compute the responses (13.43), (13.44). This is because time is continuous, and the difficulty has to do with the integrals in these formulas. For numerical calculation, one resorts to a finite sum approximation of the integrals, as we indicated in section 5.4. The Laplace transform provides an alternative approach that is exact.

The key to showing (13.43) is the fact that $e^{tA}, t \geq 0$, is the solution to the differential equation

$$\frac{d}{dt}e^{tA} = Ae^{tA}, \quad t \geq 0, \tag{13.46}$$

with initial condition $e^{0A} = I$. Note that (13.44) follows immediately from (13.43) and (13.42).

To verify (13.46) we substitute for e^{tA} from (13.45) and differentiate the sum term by term,

$$\frac{d}{dt}e^{tA} = \sum_{k=0}^{\infty} \frac{d}{dt}\frac{(tA)^k}{k!} = \sum_{k=1}^{\infty} \frac{kA}{k!}(tA)^{k-1} = A\sum_{k=1}^{\infty} \frac{(tA)^{k-1}}{(k-1)!} = Ae^{tA}.$$

We can now check that (13.43) is indeed the solution of (13.41) by taking derivatives of both sides and using (13.46):

$$\begin{aligned}
\dot{v}(t) &= Ae^{tA}v(0) + e^{0A}bx(t) + \int_0^t Ae^{(t-\tau)A}bx(\tau)d\tau \\
&= A[e^{tA}v(0) + \int_0^t Ae^{(t-\tau)A}bx(\tau)d\tau] + bx(t) \\
&= Av(t) + bx(t).
\end{aligned}$$

We turn to the main difficulty in calculating the terms on the right in the responses (13.43), (13.44), namely the calculation of the $N \times N$ matrix $e^{tA}, t \geq 0$. We determine the Laplace transform of $e^{tA}u(t), t \in Reals$, denoting it by

$$G(s) = \int_0^{\infty} e^{tA}e^{-st}dt.$$

This means that $g_{ij}(s)$ is the Laplace transform of $a_{ij}(t), t \geq 0$, denoting by $a_{ij}(t)$ and $g_{ij}(s)$ the coefficients of the $N \times N$ matrices e^{tA} and $G(s)$, respectively. The region of convergence of G, RoC, is determined later.

Using the derivative formula (13.29) in (13.46) we see that

$$sG(s) - I = AG(s),$$

which gives $G(s) = [sI - A]^{-1}$, so that the Laplace transform of $e^{tA}u(t)$ is

$$\boxed{G(s) = \int_0^\infty e^{tA}e^{-st}dt = [sI - A]^{-1}.} \tag{13.47}$$

Example 13.26: Let

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix},$$

then

$$[sI - A]^{-1} = \begin{bmatrix} s-1 & -2 \\ 2 & s-1 \end{bmatrix}^{-1} = \frac{1}{\det[sI - A]} \begin{bmatrix} s-1 & 2 \\ -2 & s-1 \end{bmatrix}.$$

The determinant is

$$\det[sI - A] = (s-1)^2 + 4 = (s-1+2i)(s-1-2i),$$

so that

$$\begin{aligned} [sI - A]^{-1} &= \begin{bmatrix} \frac{s-1}{(s-1+2i)(s-1-2i)} & \frac{2}{(s-1+2i)(s-1-2i)} \\ \frac{-2}{(s-1+2i)(s-1-2i)} & \frac{s-1}{(s-1+2i)(s-1-2i)} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1/2}{s-1+2i} + \frac{1/2}{s-1-2i} & \frac{i/2}{s-1+2i} + \frac{-i/2}{s-1-2i} \\ \frac{-i/2}{s-1+2i} + \frac{i/2}{s-1-2i} & \frac{1/2}{s-1+2i} + \frac{1/2}{s-1-2i} \end{bmatrix}. \end{aligned}$$

The region of convergence $RoC = \{s \in Complex \mid Re\{s\} > 1\}$. We find the inverse Laplace transform using table 13.3 and express it in two ways: for all $t \geq 0$,

$$\begin{aligned} e^{tA} &= e^{(1-2i)t} \begin{bmatrix} 1/2 & i/2 \\ -i/2 & 1/2 \end{bmatrix} + e^{(1+2i)t} \begin{bmatrix} 1/2 & -i/2 \\ i/2 & 1/2 \end{bmatrix} \\ &= e^{t} \begin{bmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{bmatrix}. \end{aligned}$$

The first expression shows e^{tA} as a linear combination of the exponentials $e^{(1-2i)t}$ and $e^{(1+2i)t}$, in which the exponents, $1-2i$ and $1+2i$, are the two eigenvalues of A—that is, the zeros of its characteristic polynomial, $\det[sI - A]$. The second expression shows that e^{tA} is sinusoidal with frequency 2 radians/sec equal to the imaginary part of the eigenvalues whose amplitude grows exponentially corresponding to the real part of the eigenvalues.

Returning to the general case (13.47), denote the matrix inverse of $[sI-A]$ as

$$G(s) = [sI-A]^{-1} = \frac{1}{\det[sI-A]}K(s),$$

in which $K(s)$ is the $N \times N$ matrix of co-factors of $[sI-A]$. Each coefficient $g_{ij}(s)$ of $G(s)$ is a rational polynomial of A whose denominator is the characterstic polynomial of A, $\det[sI-A]$. Therefore, if there are no pole-zero cancellations, all coefficients of $G(s)$ have the same poles—the eigenvalues of A. Because $e^{tA}u(t), t \in Reals$, is a causal signal, the region of convergence of its Laplace transform $G(s)$ is $\{s \in Complex \mid Re\{s\} > Re\{p\}\}$, in which p is the pole of G with the largest real part.

Because $\det[sI-A]$ is a polynomial of order N, G has N poles. For the system (13.41), (13.42) to be stable, the poles of $G(s)$ must have strictly negative real parts. The system of example (13.26) is unstable, because the real part of the eigenvalues is $+1$.

Suppose the characteristic polynomial has N distinct zeros $p_1, \cdots, p_N$,

$$\det[sI-A] = (s-p_1)\cdots(s-p_N).$$

Then the partial fraction expansion of $G(s)$ has the form

$$G(s) = [sI-A]^{-1} = \frac{1}{s-p_1}R_1 + \cdots + \frac{1}{s-p_N}R_N,$$

in which R_i is the matrix of residues at the pole p_i of the coefficients of $G(s)$. R_i is a constant matrix, possibly with complex coefficients, if p_i is complex. Because the inverse Laplace transform of $\frac{1}{s-p_i}$ is $e^{p_i t}u(t)$, the inverse Laplace transform of $[sI-A]^{-1}$ is

$$\boxed{e^{tA}u(t) = [e^{p_1 t}R_1 + \cdots + e^{p_N t}R_N]u(t).} \quad (13.48)$$

Thus the matrix e^{tA} as a function of t is a linear combination of $e^{p_1 t}, \cdots, e^{p_N t}$, where the p_i are the eigenvalues of A—that is the zeros of $\det[sI-A]$.

We decompose the response (13.44) into the sum of the zero-input and zero-state responses, expressing the latter as a convolution integral,

$$y(t) = c^T e^{tA} v(0) + \int_0^t h(t-\tau)x(\tau)d\tau, \ t \geq 0,$$

in which the (zero-state) impulse response is: for all $t \in Reals$,

$$h(t) = c^T e^{tA} b u(t) + d\delta(t).$$

(Here δ is the Dirac delta function.) Let $\hat{X}, \hat{Y}, \hat{H}, \hat{Y}_{zi}$ be the Laplace transforms

$$\hat{X}(s) = \int_0^\infty x(t)e^{-st}dt, \ \hat{Y}(s) = \int_0^\infty y(t)e^{-st}dt, \ \hat{H}(s) = \int_{-\infty}^\infty h(t)e^{-st}dt, \ \hat{Y}_{zi}(s) = \int_0^\infty c^T e^{tA} v(0)e^{-st}dt.$$

Then

$$\hat{Y} = \hat{H}\hat{X} + \hat{Y}_{zi},$$

in which

$$\boxed{\hat{H}(s) = c^T[sI - A]^{-1}b + d,}$$

and

$$\boxed{\hat{Y}_{zi}(s) = c^T[sI - A]^{-1}v(0).}$$

We continue with example 13.26.

Example 13.27: Suppose A is as in example 13.26, $b^T = [1 \;\; 1]^T, c^T = [2 \;\; 0]^T, d = 3$, and $v(0)^T = [0 \;\; 4]^T$. Then the transfer function is

$$\hat{H}(s) = [2 \;\; 0]\begin{bmatrix} \frac{s-1}{(s-1)^2-4} & \frac{-2}{(s-1)^2-4} \\ \frac{2}{(s-1)^2-4} & \frac{s-1}{(s-1)^2-4} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 3 = \frac{2s-6}{(s-1)^2-4} + 3,$$

and the Laplace transform of the zero-input response is

$$\hat{Y}_{zi}(s) = [2 \;\; 0]\begin{bmatrix} \frac{s-1}{(s-1)^2-4} & \frac{-2}{(s-1)^2-4} \\ \frac{2}{(s-1)^2-4} & \frac{s-1}{(s-1)^2-4} \end{bmatrix}\begin{bmatrix} 0 \\ 4 \end{bmatrix} = \frac{-16}{(s-1)^2-4}.$$

The transfer function

$$\hat{H}(s) = \frac{2s-6}{(s-1)^{-}4} + 3 = \frac{3s^2-4s-15}{s^2-2s-3}.$$

From (13.33) we know that $\hat{H}$ is also the transfer function of the differential equation

$$\frac{d^2y}{dt^2}(t) - 2\frac{dy}{dt}(t) - 3y(t) = 3\frac{d^2x}{dt^2}(t) - 4\frac{dx}{dt}(t) - 15x(t).$$

Thus this differential equation describes the same system as the state-space model of example 13.24.

This example illustrates a general way of obtaining a differential equation description of a continuous-time state-space model by means of its transfer function.

It is easier to obtain a state-space model with a specified proper transfer function,

$$\hat{H}(s) = \frac{b_{N-1}s^{N-1} + \cdots + b_1 s + b_0}{s^N + \cdots + a_1 s + a_0} + b_N.$$

(The first term in $\hat{H}$ is strictly proper. Some of the coefficients b_i, a_j may be zero.) Then the N-dimensional $[A, b, c, d]$ representation

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{N-1} \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{bmatrix}, \quad c^T = [b_0 b_1 \cdots b_{N-1}], \quad d = b_N. \tag{13.49}$$

has the same transfer function as $\hat{H}$, that is

$$c^T[sI - A]^{-1}b + d = \hat{H}(s). \tag{13.50}$$

Exercise 30 at the end of this chapter asks you to verify (13.50).

Simply by interchanging the variables s and z we see that the proper rational polynomial

$$\hat{H}(z) = \frac{b_{N-1}z^{N-1} + \cdots + b_1 z + b_0}{z^N + \cdots + a_1 z + a_0} + b_N = d + c^T[zI - A]^{-1}b$$

is the transfer function of the discrete-time $[A, b, c, d]$ representation.

Thus we can use any of three equivalent representations of LTI systems:

- difference or differential equations, used to describe many physical systems,
- transfer functions used for frequency-domain analysis, and in feedback design considered in the next chapter,
- state-space models, used in modern control theory.

13.9 Summary

The Z transform and Laplace transform have many of the same properties as the Fourier transforms. They are linear, which greatly facilitates computation of the transforms and their inverses. Moreover, the Z transform (Laplace transform) of the output of an LTI system is the product of the Z transforms (Laplace transforms) of the input and the transfer function. Thus, the Z transform (Laplace transform) plays the same role as the frequency response, describing the relationship between the input and the output as a product rather than a convolution.

Linear difference and differential equations, and state-space models of LTI systems were introduced in chapter 5 and chapter 8. However, we lacked a method to calculate the response of these models for non-zero initial conditions. The Z transform and the Laplace transform provide such a method.

Exercises

Each problem is annotated with the letter **E, T, C** which stands for exercise, requires some thought, requires some conceptualization. Problems labeled **E** are usually mechanical, those labeled **T** require a plan of attack, those labeled **C** usually have more than one defensible answer.

1. **E** Consider the signal x given by

$$\forall n, \quad x(n) = \sin(\omega_0 n)u(n).$$

(a) Show that the Z transform is

$$\forall\, z \in RoC(x), \quad \hat{X}(z) = \frac{z\sin(\omega_0)}{z^2 - 2z\cos(\omega_0) + 1},$$

where

$$RoC(x) = \{z \in Complex \mid |z| > 1\}.$$

(b) Where are the poles and zeros?

(c) Is x absolutely summable?

2. **T** Consider the signal x given by

$$\forall\, n \in Integers, \quad x(n) = a^{|n|},$$

where $a \in Complex$.

(a) Find the Z transform of x. Be sure to give the region of convergence.

(b) Where are the poles?

(c) Under what conditions is x absolutely summable?

3. **E** Consider a discrete-time LTI system with transfer function given by

$$\forall\, z \in \{z \mid |z| > 0.9\}, \quad \hat{H}(z) = \frac{z}{z - 0.9}.$$

Suppose that the input x is given by

$$\forall\, n \in Integers, \quad x(n) = \delta(n) - 0.9\delta(n-1).$$

Find the Z transform of the output y, including its region of convergence.

4. **E** Consider the exponentially modulated sinusoid y given by

$$\forall\, n \in Integers, \quad y(n) = a^{-n}\cos(\omega_0 n)u(n),$$

where a is a real number, ω_0 is a real number, and u is the unit step signal.

(a) Find the Z transform. Be sure to give the region of convergence. **Hint:** Use example 13.3 and section 13.1.6.

(b) Where are the poles?

(c) For what values of a is this signal absolutely summable?

5. **T** Suppose $x \in DiscSignals$ satisfies

$$\sum_{n=-\infty}^{\infty} |x(n)r^{-n}| < \infty, \quad 0 < r_1 < r < r_2,$$

for some real numbers r_1 and r_2 such that $r_1 < r_2$. Show that

$$\sum_{n=-\infty}^{\infty} |nx(n)r^{-n}| < \infty, \quad 0 < r_1 < r < r_2.$$

Hint: Use the fact that for any $\varepsilon > 0$ there exists $N < \infty$ such that $n(1+\varepsilon)^{-n} < 1$ for all $n > N$.

6. **T** Consider a causal discrete-time LTI system where the input x and output y are related by the difference equation

$$\forall\, n \in Integers, \quad y(n) + b_1 y(n-1) + b_2 y(n-2) = a_0 x(n) + a_1 x(n-1) + a_2 x(n-2),$$

where b_1, b_2, a_0, a_1, and a_2 are real-valued constants.

   (a) Find the transfer function.
   (b) Say as much as you can about the region of convergence.
   (c) Under what conditions is the system stable?

7. **E** This exercise verifies the time delay property of the Laplace transform. Show that if x is a continuous-time signal, τ is a real constant, and y is given by

$$\forall\, t \in Reals, \quad y(t) = x(t - \tau),$$

then its Laplace transform is

$$\forall\, s \in RoC(y), \quad \hat{Y}(s) = e^{-s\tau}\hat{X}(s),$$

with region of convergence

$$RoC(y) = RoC(x).$$

8. **E** This exercise verifies the convolution property of the Laplace transform. Suppose x and h have Laplace transforms $\hat{X}$ and $\hat{H}$. Let y be given by

$$\forall\, t \in Reals, \quad y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

Then show that the Laplace transform is

$$\forall s \in RoC(y), \quad \hat{Y}(s) = \hat{X}(s)\hat{H}(s),$$

with

$$RoC(y) \supset RoC(x) \cap RoC(h).$$

9. **T** This exercise verifies the conjugation property of the Laplace transform, and then uses this property to demonstrate that for real-valued signals, poles and zeros come in complex-conjugate pairs.

   (a) Let x be a complex-valued continuous-time signal and y be given by

$$\forall\, t \in Reals, \quad y(t) = [x(t)]^*.$$

   Show that

$$\forall\, s \in RoC(y), \quad \hat{Y}(s) = [\hat{X}(s^*)]^*,$$

   where

$$RoC(y) = RoC(x).$$

(b) Use this property to show that if x is real, then complex poles and zeros occur in complex conjugate pairs. That is, if there is a zero at $s = q$, then there must be a zero at $s = q^*$, and if there is a pole at $s = p$, then there must also be a pole at $s = p^*$.

10. **T** This exercise verifies the time scaling property of the Laplace transform. Let y be defined by
$$\forall\, t \in Reals, \quad y(t) = x(ct),$$
for some real number c. Show that
$$\forall\, s \in RoC(y), \quad \hat{Y}(s) = \hat{X}(s/c)/|c|,$$
where
$$RoC(y) = \{s \mid s/c \in RoC(x)\}.$$

11. **E** This exercise verifies the exponential scaling property of the Laplace transform. Let y be defined by
$$\forall\, t \in Reals, \quad y(t) = e^{at}x(t),$$
for some complex number a. Show that
$$\forall\, s \in RoC(y), \quad \hat{Y}(s) = \hat{X}(s-a),$$
where
$$RoC(y) = \{s \mid s-a \in RoC(x)\}.$$

12. **T** Consider a discrete-time LTI system with impulse response
$$\forall n, \quad h(n) = a^n \cos(\omega_0 n) u(n),$$
for some $\omega_0 \in Reals$. Show that if the input is
$$\forall\, n \in Integers, \quad x(n) = e^{i\omega_0 n} u(n),$$
then the output y is unbounded.

13. **E** Find and plot the inverse Z transform of
$$\hat{X}(z) = \frac{1}{(z-3)^3}$$
with

(a) $Roc(x) = \{z \in Complex \mid |z| > 3\}$

(b) $Roc(x) = \{z \in Complex \mid |z| < 3\}$.

14. **E** Obtain the partial fraction expansions of the following rational polynomials. First divide through if necessary to get a strictly proper rational polynomial.

(a)
$$\frac{z+2}{(z+1)(z+3)}$$

(b)

$$\frac{(z+2)^2}{(z+1)(z+3)}$$

(c)

$$\frac{z+2}{z^2+4}.$$

15. **E** Find the inverse Z transform x for each of the three possible regions of convergence associated with

$$\hat{X}(z) = \frac{(z+2)^2}{(z+1)(z+3)}.$$

For which region of convergence is x causal? For which is x strictly anti-causal? For which is x two-sided?

16. **E** Find the inverse Z transform x for each of the two possible regions of convergence associated with

$$\hat{X}(z) = \frac{z+2}{z^2+4}.$$

17. **E** Consider a stable system with impulse response

$$h(n) = (0.5)^n x(n).$$

Find the steady-state response to a unit step input.

18. **E** Let $h(n) = 2^n u(-n)$, all n, and $g(n) = 0.5^n u(n)$, for all n. Find $h * u$ and $g * u$, where u is the unit step.

19. This exercise shows how we can determine the transfer function and frequency response of an LTI system from its step response. Suppose a causal system with step input $x = u$, produces the output

$$\forall\, n \in Integers, \quad y(n) = (1 - 0.5^n)u(n).$$

   (a) Find the transfer function (including its region of convergence).
   (b) If the system is stable, find its frequency response.
   (c) Find the impulse response of the system.

20. Consider an LTI system with impulse response h given by

$$\forall\, n \in Integers, \quad h(n) = 2^n u(n).$$

   (a) Find the transfer function, including its region of convergence.
   (b) Use the transfer function to find the Z transform of the step response.
   (c) Find the inverse transform of the result of part (b) to obtain the step response in the time domain.

21. **E** Determine the zero-input and zero-state responses, and the transfer function for the following. In both cases take $y(-1) = y(-2) = 0$ and $x(n) = u(n)$.

    (a) $y(n) + y(n-2) = x(n), n \geq 0$.

    (b) $y(n) + 2y(n-1) + y(n-2) = x(n), n \geq 0$.

22. **E** Determine the zero-input and the zero-state responses for the following.

    (a) $5\dot{y} + 10y = 2x, y(0) = 2, x(t) = u(t)$.

    (b) $\ddot{y} + 5\dot{y} + 6y = -4x - 3\dot{x}, y(0) = -1, \dot{y}(0) = 5, x(t) = e^{-t}u(t)$.

    (c) $\ddot{y} + 4y = 8x, y(0) = 1, \dot{y}(0) = 2, x(t) = u(t)$.

    (d) $\ddot{y} + 2\dot{y} + 5y = \dot{x}, y(0) = 2, \dot{y}(0) = 0, x(t) = e^{-t}u(t)$.

23. **E** Show that the $[A, b, c, d]$ representation in example 13.23 is correct. Then show that the transfer function of the state-space model is the same as that of the difference equation.

24. **T** Consider the circuit of figure 13.5. The input is the voltage x, the output is the capacitor voltage v. The inductor current is called i.

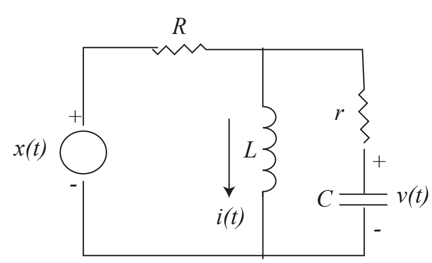

Figure 13.5: Circuit of problem 24

    (a) Derive the $[A, b, c, d]$ representation for this system using $s(t) = [i(t), v(t)]^T$ as the state.

    (b) Obtain an $[F, g, h, k]$ representation for a discrete-time model of the same circuit by sampling at times $kT, k = 0, 1, \cdots$ and using the approximation $\dot{s}(kT) = 1/T(s((k+1)T) - s(kT))$. (This is called a forward-Euler approximation.)

25. **E** For the matrix A in example 13.24, determine $e^{tA}, t \geq 0$.

26. **E** For the matrix A in example 13.26, determine $A^n, n \geq 0$.

27. **T** A continuous-time SISO system has $[A, b, c, d]$ representation with

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix},$$

in which a, b are real constants.

    (a) Find the eigenvalues of A.

    (b) For what values of a, b is the SISO system stable?

(c) Calculate $e^{tA}, t \geq 0$.

(d) Suppose $b = c = [1 \;\; 0]^T$, and $d = 0$. Find the transfer function.

28. **T** Let A be an $N \times N$ matrix. Let p be an eigenvalue of A. An N-dimensional (column) vector e, possibly complex-valued, is said to be an **eigenvector** of A corresponding to p if $e \neq 0$ and $Ae = pe$. Note that an eigenvector always exists since $\det[pI - A] = 0$. Find eigenvectors for each of the two eigenvalues of the matrices in examples 13.24 and 13.26.

29. **E** Let A be a square matrix with eigenvalue p and corresponding eigenvector e. Determine the response of the following.

    (a) $s(k+1) = As(k), k \geq 0;\; s(0) = e$.

    (b) $\dot{s}(t) = As(t), t \geq 0;\; s(0) = e$.

    Hint. Show that $A^n e = p^n e$ and $e^{tA} e = e^{pt} e$.

30. **T** Verify (13.50). Hint. First show that

$$[sI - A]^{-1} b = \frac{1}{s^N + a_{N-1} s^{N-1} + \cdots + a_0} \begin{bmatrix} 1 \\ s \\ \cdots \\ s^{N-1} \end{bmatrix},$$

by multiplying both sides by $[sI - A]$. Then check (13.50).

Chapter 14

Composition and Feedback Control

A major theme of this book is that interesting systems are often compositions of simpler systems. Systems are functions, so their composition is function composition, as discussed in section2.1.5. However, systems are often not directly described as functions, so function composition is not the easiest tool to use to understand the composition. We have seen systems described as state machines, frequency responses, and transfer functions. In chapter4 we obtained the state machine of the composite system from its component state machines. In section8.5 we obtained the frequency response of the composite system from the frequency response of its component linear time-invariant (LTI) systems. We extend the latter study in this chapter to the composition of LTI systems described by their transfer functions. This important extension allows us to consider unstable systems whose impulse response has a Z or Laplace transform, but not a Fourier transform.

As before, feedback systems prove challenging. A particularly interesting issue is how to maintain stability, and how to construct stable systems out of unstable ones. We will find that some feedback compositions of stable systems result in unstable systems, and conversely, some compositions of unstable systems result in stable systems. For example, we can stabilize the helicopter in example 12.2 using feedback, in fact we can precisely control its orientation, despite the intrinsic instability. The family of techniques for doing this is known as **feedback control**. This chapter serves as an introduction to that topic. Feedback control can also be used to drive stable systems, in which case it serves to improve their response. For example, feedback can result in faster or more precise responses, and can also prevent overshoot, where a system overreacts to a command.

We will consider three styles of composition, **cascade composition**, **parallel composition**, and **feedback composition**. In each case, two LTI systems with transfer functions $\hat{H}_1$ and $\hat{H}_2$ are combined to get a new system. The transfer functions $\hat{H}_1$ and $\hat{H}_2$ are the (Z or Laplace) transforms of the respective impulse responses, h_1 and h_2. Much of our discussion applies equally well whether the system is a continuous-time system or a discrete-time system, so in many cases we leave this unspecified.

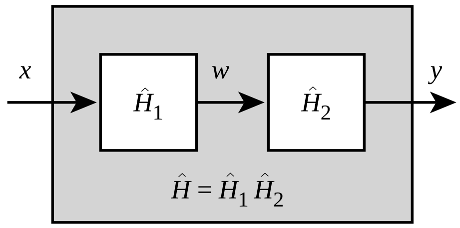

Figure 14.1: Cascade composition of two LTI systems with transfer functions H_1 and H_2.

14.1 Cascade composition

Consider the **cascade composition** shown in figure 14.1. The composition is the grey box, and it has transfer function

$$\hat{H} = \hat{H}_1 \hat{H}_2.$$

Notice that because of this simple form, if we know the pole and zero locations of the component systems, then it is easy to determine the pole and zero locations of the composition. Unless a pole of one is cancelled by a zero of the other, the poles and zeros of the composition are simply the aggregate of the poles and zeros of the components. Moreover, any pole of $\hat{H}$ must be a pole of either $\hat{H}_1$ or $\hat{H}_2$, so if $\hat{H}_1$ and $\hat{H}_2$ are both stable, then so is $\hat{H}$.

14.1.1 Stabilization

The possibility for pole-zero cancellation suggests that cascade composition might be used to stabilize an unstable system.

Example 14.1: Consider a discrete-time system with transfer function

$$\forall z \in \{z \mid |z| > |1.1|\}, \quad \hat{H}_1(z) = \frac{z}{z - 1.1} .$$

This is a proper rational polynomial with a region of convergence of the form for a causal signal, so it must be a causal system. However, it is not stable, because the region of convergence does not include the unit circle.

To stabilize this system, we might consider putting it in cascade with

$$\forall z \in Complex, \quad \hat{H}_2(z) = \frac{z - 1.1}{z} .$$

This is a causal and stable system. The transfer function of the cascade composition is

$$\hat{H}(z) = \frac{z}{z - 1.1} \frac{z - 1.1}{z} = 1 .$$

The pole at $z = 1.1$ has been cancelled, and the resulting region of convergence is the entire complex plane. Thus, the cascade composition is a causal and stable system, and we can recognize from table 13.1 that the impulse response is $h(n) = \delta(n)$.

Stabilizing systems by cancelling their poles in a cascade composition, however, is almost never a good idea. If the pole is not precisely cancelled, then no matter how small the error, the resulting system is still unstable.

Example 14.2: Suppose that in the previous example the pole location is not known precisely, and turns out to be at $z = 1.1001$ instead of $z = 1.1$. Then the cascade composition has transfer function

$$\hat{H}(z) = \frac{z}{z-1.1001}\,\frac{z-1.1}{z} = \frac{z-1.1}{z-1.1001}\,,$$

which is unstable.

14.1.2 Equalization

While cascade compositions do not usually work well for stabilization, they do often work well for **equalization**. An **equalizer** is a **compensator** that reverses **distortion**. The source of the distortion, which is often called a **channel**, must be an LTI system, and the equalizer is composed in cascade with it. At first sight this is easy to do. If the channel has transfer function $\hat{H}_1$, then the equalizer could have transfer function

$$\hat{H}_2 = \hat{H}_1^{-1},$$

in which case the cascade composition will have transfer function

$$\hat{H} = \hat{H}_1\hat{H}_2 = 1,$$

which is certainly distortion-free.

Example 14.3: Some acoustic environments for audio have **resonances**, where certain frequencies are enhanced as the sound propagates through the environment (see lab C.8 for an example). This will typically occur if the physics of the acoustic environment results in a transfer function with poles near the unit circle (for a discrete-time model) or near the imaginary axis (for a continuous-time model). Suppose for example that the acoustic environment is well modeled by a discrete-time LTI system with transfer function

$$\forall\, z \in \{z \mid |z| > 0.95\}, \quad \hat{H}_1(z) = \frac{z^2}{(z-a)(z-a^*)}\,,$$

where $a = 0.95e^{i\omega_1}$ for some frequency ω_1. Using the methods of section 13.2, we can infer that the magnitude response will have a strong peak at frequencies ω_1 and $-\omega_1$, because the positions on the unit circle $e^{i\omega_1}$ and $e^{-i\omega_1}$ are very close to the poles. This will result in distortion of the audio signal, where frequencies near ω_1 will be amplified.

An equalizer that will compensate for this distortion has transfer function

$$\hat{H}_2(z) = [\hat{H}_1(z)]^{-1} = \frac{(z-a)(z-a^*)}{z^2} = \frac{z^2 - 2Re\{a\}z + |a|^2}{z^2}\,.$$

As in example 13.2, we can recognize this as the Z transform of an FIR filter with impulse response

$$\forall\, n \in Integers, \quad h_2(n) = \delta(n) - 2Re\{a\}\delta(n-1) + |a|^2\delta(n-2).$$

This filter is causal and stable, and hence can serve as an effective equalizer.

There are a number of potential problems with this approach, however. First, the transfer function of the channel is probably not known, or at least not known precisely. Second, the channel may not have a stable and causal inverse.

Let us first examine the first difficulty, that the channel may not be known (precisely). If the channel model $\hat{H}_1$ and its inverse $\hat{H}_2$ are both stable, then the cascade composition is at least assured of being stable, even if the channel has been misconstrued. Moreover, if the equalizer is close to the inverse of the true channel, then often the distortion is significantly reduced despite the errors (see exercise 1).

This difficulty can sometimes be dealt with by adaptively varying the equalizer based on measurements of the distortion. One way to measure the distortion is to send through the channel a known sequence called a **training sequence** and observe the output of the channel. Suppose that the training sequence is a signal x with Z transform $\hat{X}$, and that the channel $\hat{H}_1$ is unknown. If we can observe the output y of the channel, and calculate its Z transform $\hat{Y}$, then the channel transfer function is simply

$$\forall\, z \in RoC(h_1), \quad \hat{H}_1(z) = \frac{\hat{Y}(z)}{\hat{X}(z)} ,$$

where $RoC(h_1)$ is determined by identifying the poles and zeros of the rational polynomial $\hat{Y}(z)/\hat{X}(z)$ and finding the one ring-shaped region that includes the unit circle and is bordered by poles. This results in a stable channel model.

Training sequences are commonly used in digital communication systems, where, for example, a radio channel introduces distortion. However, it is also common for such channels to change over time. Radio channels, for example, change if either the transmitter or receiver moves, or if the weather changes, or if obstacles appear or disappear. Repeatedly transmitting training sequences is an expensive waste of radio bandwidth, and fortunately, is not usually necessary, as illustrated in the following example.

Example 14.4: Consider a digital communication system where the channel is modeled as a discrete-time LTI system with transfer function $\hat{H}_1$, representing for example a radio transmission subsystem. Suppose that this digital communication system transmits a bit sequence represented as a discrete-time signal x of form

$$x\colon Integers \to \{0, 1\}.$$

Suppose further that we use a training sequence to obtain an initial estimate $\hat{H}_2$ of the inverse of the channel. But over time, the channel drifts, so that $\hat{H}_2$ is no longer the inverse of $\hat{H}_1$. Assuming the drift is relatively slow, then after a short time, $\hat{H}_2$ is still close to the inverse of $\hat{H}_1$, in that the cascade $\hat{H}_1\hat{H}_2$ yields only mild distortion.

That is, if $x(n) = 0$ for some n, then $y(n) \approx 0$. Similarly, if $x(n) = 1$ for some n, then $y(n) \approx 1$. Thus, we can quantize y, getting an accurate estimate x without it having to be a known training sequence. That is, when $y(n) \approx 0$, we assume that $x(n) = 0$, and when $y(n) \approx 1$, we assume that $x(n) = 1$. These assumptions are called **decisions**, and in fact, such decisions must be made anyway for digital communication to occur. We have to decide whether a 1 or a 0 was transmitted, and closeness to 1 or 0 seems like an eminently reasonable basis on which to make such a decision.

Assuming there are no errors in these decisions, we can infer that

$$\hat{H}_1 \hat{H}_2 \hat{X}_d = \hat{Y},$$

where $\hat{X}_d$ is the Z transform of the decision sequence. So, without using another training sequence, we can revise our estimate of the channel transfer function as follows,

$$\hat{H}_1 = \frac{\hat{Y}}{\hat{H}_2 \hat{X}_d}.$$

We replace our equalizer $\hat{H}_2$ with

$$\hat{H}'_2 = [\hat{H}_1]^{-1} = \frac{\hat{H}_2 \hat{X}_d}{\hat{Y}} \ .$$

Of course, we now start using $\hat{H}'_2$, which will come closer to correcting the channel distortion, which will make our decisions more reliable for the next update.

Example 14.4 outlines a widely used technique called **decision-directed adaptive equalization**. It is so widely used, in fact, that it may be found in every digital cellular telephone and almost every modem, including voiceband data modems, radio modems, cable modems, DSL modems, etc. The algorithms used in practice to update the transfer function of the equalizer are not exactly as shown in the example, and their details are beyond the scope of this text, but they follow the general principle in the example.

Let us now turn our attention to the second difficulty with equalization, that the channel may not have a stable and causal inverse. We begin with an example.

Example 14.5: Suppose that, similar to example 14.3, a channel has transfer function

$$\forall\, z\ in \{z \mid |z| > 0.95\}, \quad \hat{H}_1(z) = \frac{z}{(z-a)(z-a^*)} \ ,$$

where $a = 0.95e^{i\omega_1}$ for some frequency ω_1. The inverse is

$$[\hat{H}_1(z)]^{-1} = \frac{(z-a)(z-a^*)}{z} = \frac{z^2 - 2Re\{a\}z + |a|^2)}{z} \ ,$$

which is not a proper rational polynomial. Thus, this cannot be the Z transform of a causal signal. Implementing a non-causal equalizer will usually be impossible, since it will require knowing future inputs. However, suppose we simply force the equalizer have a proper rational polynomial transfer function by dividing by a high enough power M of z to make $[\hat{H}_1(z)]^{-1}/z^M$ proper. In this example, $M = 1$ is sufficient, so we define the equalizer to be

$$\hat{H}_2(z) = \frac{[\hat{H}_1(z)]^{-1}}{z} = \frac{z^2 - 2Re\{a\}z + |a|^2)}{z^2} \ ,$$

which we again recognize as the Z transform of an FIR filter with impulse response

$$\forall\, n \in Integers, \quad h_2(n) = \delta(n) - 2Re\{a\}\delta(n-1) + |a|^2\delta(n-2).$$

This filter is causal and stable, but does it serve as an effective equalizer? Consider now the cascade,

$$\hat{H}(z) = \hat{H}_1(z)\hat{H}_2(z) = \frac{1}{z}.$$

From section 13.1.2 we recognize this as the transfer function of the unit delay system. That is, the equalizer completely compensates for the distortion, but at the expense of introducing a one sample delay. This is usually a perfectly acceptable cost.

Example 14.5 demonstrates that when the channel inverse is not a proper rational polynomial, then introducing a delay may enable construction of a stable and causal equalizer. Not all equalization stories have such a happy ending, however. Consider the following example.

Example 14.6: Consider a channel with the following transfer function,

$$\forall\, z \in \{z \in Complex \mid z \neq 0\}, \quad \hat{H}_1(z) = \frac{z-2}{z} \ .$$

This is a stable and causal channel. Its inverse is

$$[\hat{H}_1(z)]^{-1} = \frac{z}{z-2} \ .$$

This has a pole at $z = 2$, so in order to be stable, it would have to be anti-causal (so that the region of convergence can include the unit circle). Implementing an anti-causal equalizer is usually not possible.

Example 14.6 shows that not all channels can be inverted by an equalizer. All is not lost, however. Given a channel $\hat{H}_1(z)$ that has a rational Z transform, we can usually find a transfer function $\hat{H}_2(z)$ that compensates for the magnitude response part of the distortion. That is, we can find a transfer function $\hat{H}_2(z)$ that is stable and causal such that the magnitude response of the composite satisfies

$$|H_1(\omega)H_2(\omega)| = |\hat{H}_1(e^{i\omega})\hat{H}_2(e^{i\omega})| = 1.$$

For some applications, this is sufficient. In audio equalization, for example, this is almost always sufficient, because the human ear is not very sensitive to the phase of audio signals. It hears only the magnitude of the frequency components.

Example 14.7: Continuing example 14.6, let $\hat{H}_2$ be given by

$$\hat{H}_2 = \frac{z}{1-2z} = \frac{-0.5z}{z-0.5} .$$

This has a pole at $z = 0.5$, and is a proper rational polynomial, so it can be the transfer function of a causal and stable filter. Consider the cascade composition,

$$\hat{H}(z) = \hat{H}_1(z)\hat{H}_2(z) = \frac{z-2}{z} \cdot \frac{-0.5z}{z-0.5} = \frac{1-0.5z}{z-0.5} .$$

This hardly looks like what we want, but if we rewrite it slightly, it is easy to show that the magnitude frequency response has value one for all ω,

$$\hat{H}(z) = \frac{1-0.5z}{z-0.5} = z\frac{z^{-1}-0.5}{z-0.5} .$$

The magnitude frequency response is

$$|H(\omega)| = |\hat{H}(e^{i\omega})| = |e^{i\omega}| \cdot \frac{|e^{-i\omega}-0.5|}{|e^{i\omega}-0.5|} = 1 .$$

This magnitude is equal to 1 because the numerator, $e^{-i\omega} - 0.5$, is the complex conjugate of the denominator, $e^{i\omega} - 0.5$, so they have the same magnitude.

The method in example 14.7 can be generalized so that for most channels it is possible to cancel any magnitude distortion. The key is that if the channel transfer function has a zero outside the unit circle, say at $z = a$, then its inverse has a pole at the same location, $z = a$. A pole outside the unit circle makes it impossible to have a stable and causal filter. So the trick is to place a pole instead at $z = 1/a^*$. This pole will cancel the effect on the magnitude (but not the phase) of the zero at $z = a$.

There are still channels for which this method will not work.

Example 14.8: Consider a channel given by

$$\forall\, z \in \{z \in Complex \mid z \neq 0\}, \quad \hat{H}_1(z) = \frac{z-1}{z} .$$

This has a pole at $z = 0$ and a zero at $z = 1$. Its inverse cannot be stable because it will have a pole at $z = 1$. In fact, no equalization is possible. This is intuitive because the frequency response is zero at $\omega = 0$, and no stable equalizer in cascade with this channel can reconstruct the original component at $\omega = 0$. It would have to have infinite gain at $\omega = 0$, which would make it unstable.

14.2 Parallel composition

Consider the **parallel composition** shown in figure 14.2. The transfer function of the composition system is

$$\hat{H} = \hat{H}_1 + \hat{H}_2.$$

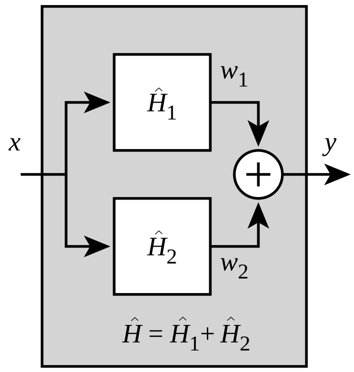

Figure 14.2: Parallel composition of two LTI systems with transfer functions H_1 and H_2.

This is valid whether these are Laplace transforms or Z transforms. Once again, notice that a pole of $\hat{H}$ must be a pole of either $\hat{H}_1$ or $\hat{H}_2$, so if $\hat{H}_1$ and $\hat{H}_2$ are stable, then so is $\hat{H}$. At the poles of $\hat{H}_1$, $\hat{H}_1(z)$ is infinite, so very likely a pole of $\hat{H}_1$ will also be a pole of $\hat{H}$. However, just as in the cascade composition, this pole may be cancelled by a zero.

Determining the location of the zeros of the composition, however, is slightly more complicated than for cascade composition. The sum has to be put into rational polynomial form, and the polynomials then need to be factored.

14.2.1 Stabilization

Just as with cascade composition, stabilizing systems by cancelling their poles in a parallel composition is possible, but is almost never a good idea.

Example 14.9: Consider a discrete-time system with transfer function

$$\forall\, z \in \{z \mid |z| > |1.1|\}, \quad \hat{H}_1(z) = \frac{z}{z-1.1}\ .$$

This describes a causal but unstable system. Suppose we combine this in parallel with a system with transfer function

$$\forall\, z \in \{z \mid |z| > |1.1|\}, \quad \hat{H}_2(z) = \frac{-1.1}{z-1.1}\ .$$

This is again causal and unstable. The transfer function of the parallel composition is

$$\hat{H}(z) = \frac{z}{z-1.1} + \frac{-1.1}{z-1.1} = \frac{z-1.1}{z-1.1} = 1\ .$$

The pole at $z = 1.1$ has been cancelled, and the resulting region of convergence is the entire complex plane. Thus, the parallel composition is a causal and stable system with impulse response $h(n) = \delta(n)$.

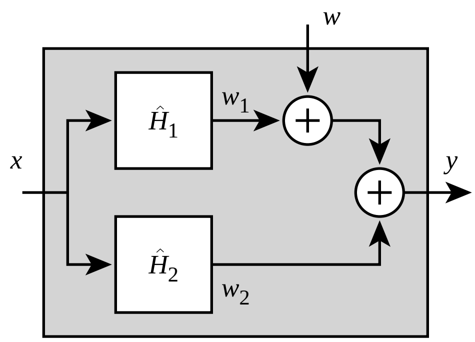

Figure 14.3: Structure of a noise canceller.

However, if the pole is not precisely cancelled, then no matter how small the error, the resulting system is still unstable.

Example 14.10: Suppose that in the previous example the pole location is not known precisely, and turns out to be at $z = 1.1001$ instead of $z = 1.1$. Then the parallel composition has transfer function

$$\hat{H}(z) = \frac{z}{z - 1.1001} + \frac{-1.1}{z - 1.1} = \frac{z^2 - 2.2z + 1.21001}{(z - 1.1001)(z - 1.1)} ,$$

which is unstable.

14.2.2 Noise cancellation

While parallel compositions do not usually work well for stabilization, with a small modification they do often work well for **noise cancellation**. A **noise canceller** is a compensator that removes an unwanted component from a signal. The unwanted component is called **noise**.

The pattern of a noise cancellation problem is shown in figure 14.3. The signal x is a noise source. This signal is filtered by $\hat{H}_1$ and added to the **desired signal** w. The result is a noisy signal. To cancel the noise, the signal from the noise source is filtered by a noise cancelling filter $\hat{H}_2$ and the result is added to the noisy signal. If x has (Laplace or Z) transform $\hat{X}$, w has transform $\hat{W}$, and y has transform $\hat{Y}$, then

$$\hat{Y} = \hat{W} + (\hat{H}_1 + \hat{H}_2)\hat{X}.$$

From this it is evident that if we choose

$$\hat{H}_2 = -\hat{H}_1,$$

then y will be a clean (noise-free) signal, equal to w. The following examples describe real-world applications of this pattern.

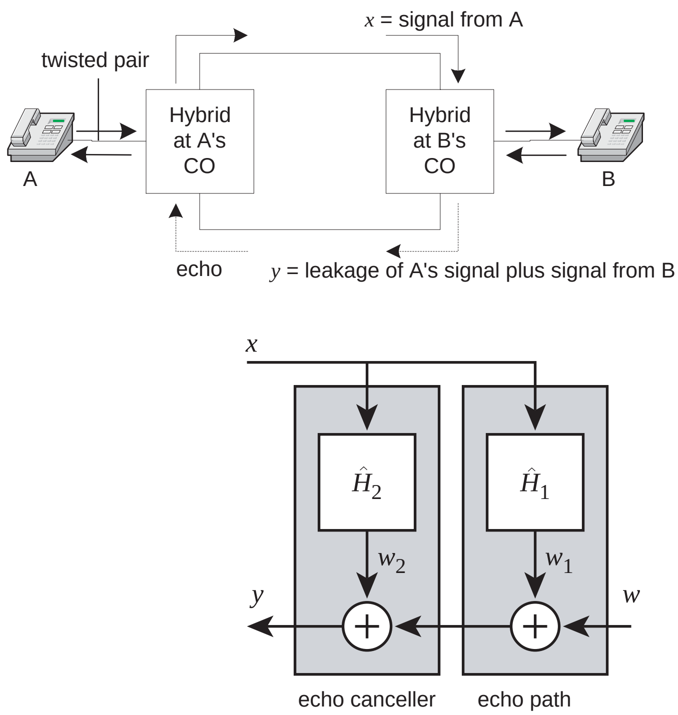

Figure 14.4: A telephone central office converts the two-wire connection with a customer telephone into a four-wire connection with the telephone network using a device called a hybrid. An imperfect hybrid leaks, causing echo. An echo canceller removes the leaked signal.

Example 14.11: A connection to the telephone network uses two wires (called a **twisted pair**, consisting of **tip and ring**) to connect a telephone to a central office. The central office may be, perhaps, 4 kilometers away. The two wires carry voice signals to and from the customer premises, representing the voice signals as a voltage difference across the two wires. Since two wires can only have one voltage difference across them, the incoming voice signal and the outgoing voice signal share the same twisted pair.

The central office needs to separate the voice signal from the local customer premises (called the **near-end signal**) from the voice signal that comes from the other end of the connection (called the **far-end signal**). The near end signal is typically digitized (sampled and quantized), and a discrete-time representation of the voice signal is transmitted over the network to the far end. The network itself consists of circuits that can carry voice signals in one direction at a time. Thus, in the network, four wires (or equivalent) are required for a telephone connection, one wire pair for each direction.

As indicated in figure 14.4, the conversion from a two-wire to a four-wire connection is done by a device called a **hybrid**.[1] A connection between subscribers *A* and *B* involves

[1] A hybrid is a Wheatstone bridge, a circuit that can separate two signals based on the electrical impedance looking into the local twisted pair and the electrical impedance looking into the network. The design of this circuit is a suitable

two hybrids, one in each subscriber's central office. The hybrid in B's central office ideally will pass all of the incoming signal x to B's two-wire circuit, and none back into the network. However, the hybrid is not perfect, and some of the incoming signal x leaks through the hybrid into the return path back to A. The signal y in the figure is the sum of the signal from B and the leaked signal from A. A hears the leaked signal as an echo, since it is A's own signal, delayed by propagation through the telephone network.

If the telephone connection includes a satellite link, then the delay from one end of the connection to the other is about 300ms. This is the time it takes for a radio signal to propagate to a geosynchronous satellite and back. The echo traverses this link twice: once going from A to B, and the second time coming back. Thus, the echo is A's own signal delayed by about 600ms. For voice signals, 600ms of delay is enough to create a very annoying echo that can make it difficult to speak. Humans have difficulty speaking when they hear their own voices 600ms later. Consequently, the designers of the telephone network have put echo cancellers in to prevent the echo from occuring.

Let $\hat{H}_1$ be the transfer function of the hybrid leakage path. The echo canceller is the filter $\hat{H}_2$ placed in parallel composition with the hybrid, as shown in the figure. The output w_2 of this filter is added to the output $w_1 + w$ of the hybrid, so the signal that actually goes back is $y = w_2 + w_1 + w$. If

$$\hat{H}_2 = -\hat{H}_1,$$

then $y = w$ and the echo is cancelled perfectly. Moreover, note that as long as $\hat{H}_1$ is stable and causal, so will be the echo canceller $\hat{H}_2$.

However, $\hat{H}_1$ is not usually known in advance, and also it changes over time. So either a fixed $\hat{H}_2$ is designed to match a 'typical' $\hat{H}_1$, or an adaptive echo canceller is designed that estimates the characteristics of the echo path ($\hat{H}_1$) and changes $\hat{H}_2$ accordingly. Adaptive echo cancellers are common in the telephone network today.

The following example combines cascade and parallel composition to achieve noise cancellation.

Example 14.12: Consider a microphone in a noisy environment. For example, a traffic helicopter might be used to deliver live traffic reports over the radio, but the (considerable) background noise of the helicopter would be highly undesirable on the radio. Fortunately, the background noise can be cancelled. Referring to figure 14.5, suppose that w is the announcer's voice, x is the engine noise, and $\hat{H}_1$ represents the acoustic path from the engine noise to the microphone. The microphone picks up both the engine noise and the announcer's voice, producing the noisy signal w_4. We can place a second microphone somewhere far enough from the announcer so as to not pick up much of his or her voice. Since this microphone is in a different place, say on the back of the announcer's helmet, the acoustic path is different, so we model that path with another transfer function $\hat{H}_2$. To cancel the noise, we design a filter $\hat{H}_3$. This filter needs to equalize (invert) $\hat{H}_2$ and cancel $\hat{H}_1$. That is, its ideal value is

$$\hat{H}_3 = -\hat{H}_1/\hat{H}_2.$$

topic for a text on electrical circuits.

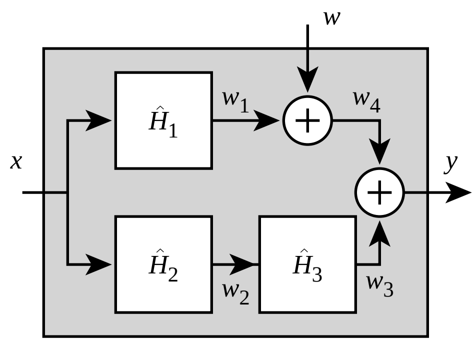

Figure 14.5: Traffic helicopter noise cancellation/equalization problem.

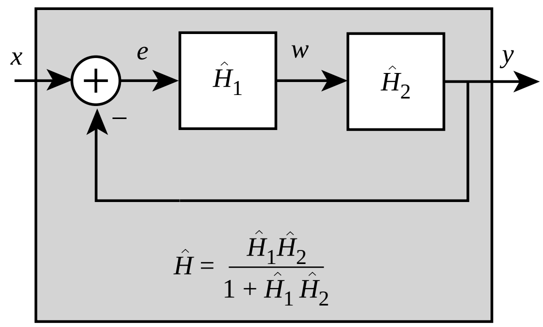

Figure 14.6: Negative feedback composition of two LTI systems with transfer functions H_1 and H_2.

Of course, as with the equalization scenario, we have to ensure that this filter remains stable. Once again, in practice, it is necessary to make the filter adaptive.

14.3 Feedback composition

Consider the **feedback composition** in figure 14.6. It is a composition of two systems with transfer functions $\hat{H}_1$ and $\hat{H}_2$. We assume that these systems are causal and that $\hat{H}_1$ and $\hat{H}_2$ are proper rational polynomials in z or s. The regions of convergence of these two transfer functions are those suitable for causal systems (the region outside the largest circle passing through a pole, for discrete time, and the region to the right of the pole with the largest real part, for continuous-time).

In terms of Laplace or Z transforms, the signals in the figure are related by

$$\hat{Y} = \hat{H}_2\hat{H}_1\hat{E},$$

and

$$\hat{E} = \hat{X} - \hat{Y}.$$

Notice that, by convention, the feedback term is subtracted, as indicated by the minus sign adjacent to the adder (for this reason, this composition is called **negative feedback**). Combining these two equations to elimintate $\hat{E}$, we get

$$\hat{Y} = \hat{H}_1\hat{H}_2(\hat{X} - \hat{Y}),$$

which we can solve for the transfer function of the composition,

$$\hat{H} = \frac{\hat{Y}}{\hat{X}} = \frac{\hat{H}_1\hat{H}_2}{1 + \hat{H}_1\hat{H}_2} \,. \tag{14.1}$$

This is often called the **closed-loop transfer function**, to contrast it with the **open-loop transfer function**, which is simply $\hat{H}_1\hat{H}_2$. We will assume that this resulting system is causal, and that the region of convergence of this transfer function is therefore determined by the roots of the denominator polynomial, $1 + \hat{H}_1\hat{H}_2$.

The closed-loop transfer function is valid as long as the denominator $1 + \hat{H}_1\hat{H}_2$ is not identically zero (that is, it is not zero for *all s* or *z* in *Complex* – it may be zero *some s* or *z* in *Complex*). This is sufficient for the feedback loop to be well-formed, although in general, this fact is not trivial to demonstrate (exercise 8 considers the easier case where $\hat{H}_1\hat{H}_2$ is causal and strictly proper, in which case the system $\hat{H}_1\hat{H}_2$ has state-determined output). We will assume henceforth, without comment, that the denominator is not identically zero.

Feedback composition is useful for stabilizing unstable systems. In the case of cascade and parallel composition, a pole of the composite must be a pole of one of the components. The only way to remove or alter a pole of the components is to cancel it with a zero. For this reason, cascade and parallel composition are *not* effective for stabilizing unstable systems. Any error in the specification of the unstable pole location results in a failed cancellation, which results in an unstable composition.

In contrast, the poles of the feedback composition are the roots of the denominator $1 + \hat{H}_1\hat{H}_2$, which are generally quite different from the poles of $\hat{H}_1$ and $\hat{H}_2$. This leads to the following important conclusion:

> The poles of a feedback composition can be different from the poles of its component subsystems. Consequently, unstable system can be effectively and robustly stabilized by feedback.

The stabilization is **robust** in that small changes in the pole or zero locations do not result in the composition going unstable. We will be able to quantify this robustness.

14.3.1 Proportional controllers

In control applications, one of the two systems being composed, say $\hat{H}_2$, is called the **plant**. This is a physical system that is given to us to control. Its transfer function is determined by its physics. The second system being composed, say $\hat{H}_1$, is the **controller**. We design this system to get the plant to do what we want. The following example illustrates a simple strategy called a **proportional controller** or **P controller**.

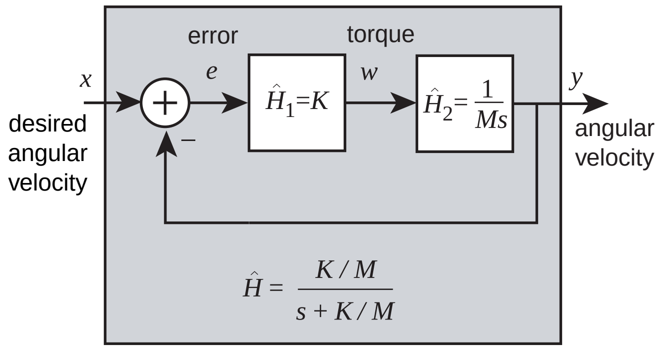

Figure 14.7: A negative feedback proportional controller with gain K.

Example 14.13: For this example we take as the plant the simplified continuous-time helicopter model of example 12.2,

$$\dot{y}(t) = \frac{1}{M} w(t).$$

Here $y(t)$ is the angular velocity at time t and $w(t)$ is the torque. M is the moment of inertia.

We have renamed the input w (instead of x) because we wish to control the helicopter, and the control input signal will not be the torque. Instead, let's define the input x to be the desired angular velocity. So, to get the helicopter to not rotate, we provide input $x(t) = 0$.

Let us call the impulse response of the plant h_2, to conform with the notation in figure 14.6; it is given by

$$\forall\, t \in Reals, \quad h_2(t) = u(t)/M,$$

where u is the unit step. The transfer function is $\hat{H}_2(s) = 1/(Ms)$, with $RoC(h) = \{s \in Complex \mid Re(s) > 0\}$. $\hat{H}_2$ has a pole at $s = 0$, so this is an unstable system.

As a compensator we can simply place a gain K in a negative feedback composition, as shown in figure 14.7. The intuition is as follows. Suppose we wish to keep the helicopter from rotating. That is, we would like the output angular velocity to be zero, $y(t) = 0$. Then we should apply an input of zero, $x(t) = 0$. However, the plant is unstable, so even with a zero input, the output diverges (even the smallest non-zero initial condition or the smallest input disturbance will cause it to diverge). With the feedback arrangement in figure 14.7, if the output angular velocity rises above zero, then the input is modified downwards (the feedback is negative), which will result in a negative torque being applied to the plant, which will counter the rising velocity. If the output angular velocity drops below zero, then the torque will be modified upwards, which again will tend to counter the dropping velocity. The output velocity will stabilize at zero.

To get the helicopter to rotate, for example to execute a turn, we simply apply a non-zero input. The feedback system will again compensate so that the helicopter will rotate at the angular velocity specified by the input.

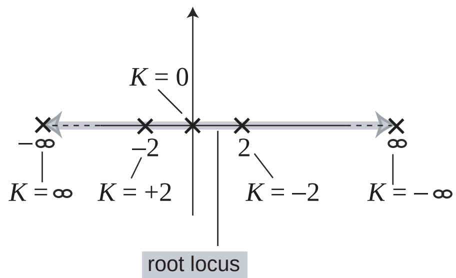

Figure 14.8: Root locus of the helicopter P controller.

The signal e is the difference between the input x, which is the desired angular velocity, and the output y, which is the actual angular velocity. It is called the **error signal**. Intuitively, this signal is zero when everything is as desired, when the output angular velocity matches the input.

A compensator like that in example 14.13 and figure 14.7 is called a **proportional controller** or **P controller**. The input w to the plant is proportional to the error e. The objective of the control system is to have the output y of the plant track the input x as closely as possible. I.e., the error e needs to be small. We can use (14.1) to find the transfer function of the closed-loop system.

Example 14.14: Continuing with the helicopter of example 14.13, the closed loop system transfer function is

$$\hat{G}(s) = \frac{K\hat{H}(s)}{1+K\hat{H}(s)} = \frac{K/M}{s+K/M} \ . \tag{14.2}$$

which has a pole at $s = -K/M$. If $K > 0$, the closed loop system is stable, and if $K < 0$, it is unstable. Thus, we have considerable freedom to choose K. How should we choose its value?

As K increases from 0 to ∞, the pole at at $s = -K/M$ moves left from 0 to $-\infty$. As K decreases from 0 to $-\infty$, the pole moves to the right from 0 to ∞. The locus of the pole as K varies is called the **root locus**, since the pole is a root of the denominator polynomial.

Figure 14.8 shows the root locus as a thick gray line, on which are marked the locations of the pole for $K = 0, \pm 2, \pm\infty$. Since there is only one pole, the root locus comprises only one 'branch'. In general the root locus has as many branches as the number of poles, with each branch showing by the movement of one pole as K varies.

Note that in principle, the same transfer function as the closed-loop transfer function can be achieved by a cascade composition. But as in example 14.1, the resulting system is not robust, in that even the smallest change in the pole location of the plant can cause the system to go unstable (see problem 6). The feedback system, however, is robust, as shown in the following example

Example 14.15: Continuing with the P controller for the helicopter, suppose that our model of the plant is not perfect, and its actual transfer function is

$$\hat{H}_2(s) = \frac{1}{M(s-\varepsilon)} ,$$

for some small value of $\varepsilon > 0$. In that case, the closed loop transfer function is

$$\hat{H}(s) = \frac{K/M}{s-\varepsilon+K/M} ,$$

which has a pole at $s = \varepsilon - K/M$. So the feedback system remains stable so long as

$$\varepsilon < K/M.$$

In practice, when designing feedback controllers, we first quantify our uncertainty about the plant, and then determine the controller parameters so that under all possible plant transfer functions, the closed-loop system is stable.

Example 14.16: Continuing the helicopter example, we might say that $\varepsilon < 0.5$. In that case, if we choose K so that $K/M > 0.5$, we would guarantee stability for all values of $\varepsilon < 0.5$. We then say that the proportional feedback controller is **robust** for all plants with $\varepsilon < 0.5$.

We still have a large number of choices for K. How do we select one? To understand the implications of different choices of K we need to study the behavior of the output y (or the error signal e) for different choices of K. In the following examples we use the closed-loop transfer function to analyze the response of a proportional controller system to various inputs. The first example studies the response to a step function input.

Example 14.17: Continuing the helicopter example, suppose that the input is a step function, $\forall t, x(t) = au(t)$ where a is a constant and u is the unit step. This input declares that at time $t = 0$, we wish for the helicopter to begin rotating with angular velocity a. The closed-loop transfer function is given by (14.2), and the Laplace transform of x is $\hat{X}(s) = a/s$, from table 13.3, so the Laplace transform of the output is

$$\hat{Y}(s) = \hat{G}(s)\hat{X}(s) = \frac{K/M}{s+K/M} \cdot \frac{a}{s}$$

Carrying out the partial fraction expansion, this becomes

$$\hat{Y}(s) = \frac{-a}{s+K/M} + \frac{a}{s} .$$

We can use this to find the inverse Laplace transform,

$$\forall t, \quad y(t) = -ae^{-Kt/M}u(t) + au(t).$$

The second term is the **steady-state response** y_{ss}, which in this case equals the input. So the first term is the **tracking error** y_{tr}, which goes to zero faster for larger K. Hence for step inputs, the larger the gain K, the smaller the tracking error.

In the previous example, we find that the error goes to zero when the input is a step function. Moreover, the error goes to zero faster if the gain K is larger than if it is smaller. In the following example, we find that if the input is sinusoidal, then larger gain K results in an ability to track higher frequency inputs.

Example 14.18: Suppose the input to the P controller helicopter system is a sinusoid of amplitude A and frequency ω_0,

$$\forall\, t \in Reals, \quad x(t) = A(\cos \omega_0 t)u(t).$$

We know that the response can be decomposed as $y = y_{tr} + y_{ss}$. The transient response y_{tr} is due to the pole at $s = -K/M$, and so it is of the form

$$\forall\, t \in Reals, \quad y_{tr}(t) = Re^{-Kt/M},$$

for some constant R. The steady-state response is determined by the frequency response at ω_0. The frequency response is

$$\forall\, \omega \in Reals, \quad H(\omega) = \hat{H}(i\omega) = \frac{K/M}{i\omega + K/M}\ ,$$

with magnitude and phase given by

$$|H(\omega)| = \frac{K/M}{[\omega^2 + (K/M)^2]^{1/2}}, \quad \angle H(\omega) = -\tan^{-1}\frac{\omega M}{K}.$$

So the steady-state response is

$$\forall t, \quad y_{ss}(t) = |H(\omega_0)| A \cos(\omega_0 t + \angle H(\omega_0)).$$

Thus the steady-state output is a sinusoid of the same frequency as the input but with a smaller amplitude (unless $\omega_0 = 0$). The larger ω_0 is, the smaller the output amplitude. Hence, the ability of the closed-loop system to track a sinusoidal input decreases as the frequency of the sinusoidal input increases. However, increasing K reduces this effect. Thus, larger gain in the feedback loop improves its ability to track higher frequency sinusoidal inputs.

In addition to the reduction in amplitude, the output has a phase difference. Again, if $\omega_0 = 0$, there is no phase error, because $\tan^{-1}(0) = 0$. As ω_0 increases, the phase lag increases (the phase angle decreases). Once again, however, increasing the gain K reduces the effect.

The previous two examples suggest that large gain in the feedback loop is always better. For a step function input, it causes the transient error to die out faster. For a sinusoidal input, it improves the ability to track higher frequency inputs, and it reduces the phase error in the tracking. A large gain is not always a good idea, however, as seen in the next example, a DC motor.

Example 14.19: The angular position y of a DC motor is determined by the input voltage w according to the differential equation

$$M\ddot{y}(t) + D\dot{y}(t) = Lw(t),$$

where M is the moment of inertia of the rotor and attached load, $D\dot{y}$ is the damping force, and the torqe Lw is proportional to the voltage. The transfer function is

$$\hat{H}_2(s) = \frac{L}{Ms^2 + Ds} = \frac{L/M}{s(s + D/M)} .$$

which has one pole at $s = 0$ and one pole at $s = -D/M$. By itself the DC motor is unstable because of the pole at $s = 0$. The transfer function of the feedback composition with proportional gain K is

$$\hat{H}(s) = \frac{K\hat{H}_2(s)}{1 + K\hat{H}_2(s)} = \frac{KL}{Ms^2 + Ds + KL} .$$

There are two closed loop poles—the roots of $Ms^2 + Ds + KL$—located at

$$s = -\frac{D}{2M} \pm \sqrt{\frac{D^2}{4M^2} - \frac{KL}{M}} .$$

The closed loop system is stable if both poles have negative real parts, which is the case if $K > 0$. If $K < D^2/(4ML)$ both poles are real. But if $K > D^2/(4ML)$, the two poles form a complex conjugate pair located at

$$s = -\frac{D}{2M} \pm i\sqrt{\frac{KL}{M} - \frac{D^2}{4M^2}} .$$

The real part is fixed at $D/2M$, but the imaginary part increases with K. We investigate performance for the parameter values $L/M = 10, D/M = 0.1$. The transfer function is

$$\hat{H}(s) = \frac{10K}{s^2 + 0.1s + 10K}.$$

Because there are two poles, the root locus has two branches, as shown in figure 14.9. For $K = 0$, the two poles are located at 0 and -0.1, as illustrated by crosses in the figure. As K increases the two poles move towards each other, coinciding at -0.05 when $K = 0.00025$. For larger values of K, the two branches split into a pair of complex conjugate poles.

To appreciate what values of $K > 0$ to select for good tracking, we consider the response to a step input $x = u(t)$ for two different values of K. For $K = 0.00025$, the Laplace transform of the output y is

$$\hat{Y}(s) = \frac{10K}{s^2 + 0.1s + 10K}\frac{1}{s} = \frac{0.0025}{(s + 0.05)^2}\frac{1}{s} = -\frac{1}{s + 0.05} - \frac{0.05}{(s + 0.05)^2} + \frac{1}{s} .$$

So the time domain response is

$$\forall t, \quad y(t) = \{-e^{-0.05t} - 0.05te^{-0.05t}\}u(t) + u(t). \qquad (14.3)$$

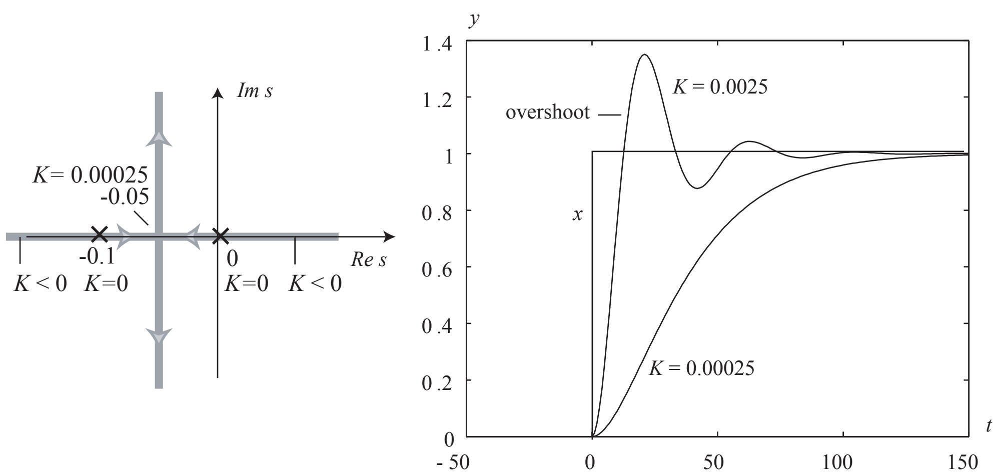

Figure 14.9: The root locus and step response for two values of K of a DC motor with proportional feedback.

For $K = 0.0025$, the Laplace transform of the output y is

$$\hat{Y}(s) = \frac{0.025}{s^2 + 0.1s + 0.025}\frac{1}{s} \approx \frac{-0.5 + i0.17}{s + 0.05 - i0.15} + \frac{-0.5 - 0.17i}{s + 0.05 + i0.15} + \frac{1}{s}.$$

So

$$\begin{aligned} \forall t, \quad y(t) &= e^{-0.5t}[0.527e^{i(0.15t+2.82)} + 0.527e^{-i(0.15t+2.82)}]u(t) + u(t) \\ &= 0.527e^{-0.5t} \times 2\cos(0.15t + 2.82)u(t) + u(t). \end{aligned} \tag{14.4}$$

The right-hand panel in figure 14.9 shows plots of the responses (14.3) and (14.4) that illustrate the design tradeoffs. In both cases, the output approaches the input as $t \to \infty$, so the asymptotic tracking error is zero. The response for the higher gain is faster but it overshoots the asymptotic value. The response for the lower gain is slower but there is no overshoot. In this example, K must be selected to balance speed of response versus the magnitude of the overshoot. In some applications, overshoot may be completely unacceptable.

We can now investigate the proportional feedback control in a general setting. Suppose the plant transfer function is a proper rational polynomial

$$\hat{H}_2(s) = \frac{\hat{A}(s)}{\hat{B}(s)},$$

where $\hat{A}$ has degree M, $\hat{B}$ has degree N, and $M \leq N$ ($\hat{H}_2$ is proper). The closed loop transfer function is

$$\hat{H}(s) = \frac{K\hat{H}_2(s)}{1 + K\hat{H}_2(s)} = \frac{K\hat{A}(s)}{\hat{B}(s) + K\hat{A}(s)}\,. \tag{14.5}$$

The closed loop poles are the N roots of the equation $\hat{B}(s)+K\hat{A}(s)=0$. These roots will depend on K, so we denote them $p_1(K),\cdots,p_n(K)$. As K varies, these roots will trace out the N branches of the root locus. At $K=0$, the poles are the roots of $\hat{B}(s)=0$, which are the poles of the plant transfer function $\hat{A}(s)/\hat{B}(s)$. The stability of the closed loop plant requires that K must be such that

$$Re\{p_1(K)\}<0,\cdots,Re\{p_n(K)\}<0. \tag{14.6}$$

Within those values of K that satisfy (14.6) we must select K to get a good response.

The following example shows that a proportional compensator may be unable to guarantee closed loop stability.

Example 14.20: Consider a plant transfer function given by

$$\hat{H}_2(s)=\frac{\hat{A}(s)}{\hat{B}(s)}=\frac{s+1}{(s-1)(s^2+0.5s+1.25)}\ .$$

There is one zero at $s=-1$, one pole at $s=1$ and a pair of complex conjugate poles at $s=-0.5\pm i1.09$. The plant is unstable because of the pole at $s=1$. The closed loop poles are the roots of the polynomial

$$P(K,s)=K\hat{A}(s)+\hat{B}(s)=K(s+1)+(s-1)((s+0.25)^2+1.188).$$

Figure 14.10 shows the three branches of the root locus plot for $K>0$. As K increases, the unstable pole moves towards the zero, while the complex conjugate poles move into the right-half plane. We need to find the values of K that satisfy the stability condition (14.6). The value of K for which the pole at $s=1$ moves to $s=0$ is obtained from the condition $P(K,0)=0$, which gives $K-(0.5^2+1)=0$ or $K=1.25$. So one condition for stability is $K>1.25$. The complex conjugate poles cross the imaginary axis at $s=\pm i1.15$ for $K=0.6$. So the second condition for stability is $K<0.6$. The two conditions $K>1.25$ and $K<0.6$ are inconsistent, so no proportional compensator can stabilize this system.

We return to the general discussion. Suppose the stability condition (14.6) can be met. Among the values of K that achieve stability, we select that value for which the output y closely tracks a step input, $x=u$. In this case, the Laplace transform of the input is $1/s$, so the Laplace transform of y is, from (14.5),

$$\hat{Y}(s)=\hat{H}(s)\frac{1}{s}=\frac{K\hat{A}(s)}{\hat{B}(s)+K\hat{A}(s)}\frac{1}{s}\ . \tag{14.7}$$

Assuming for simplicity that all the poles $p_1(K),\cdots,p_n(K)$ have multiplicity 1, $\hat{Y}$ has the partial fraction expansion

$$\hat{Y}(s)=\sum_{i=1}^{n}\frac{R_i}{s-p_i(K)}+\frac{R_0}{s}\ ,$$

and hence the time-domain behavior of

$$\forall t,\quad y(t)=\left[\sum_{i=1}^{n}R_ie^{p_i(K)t}\right]u(t)+R_0u(t).$$

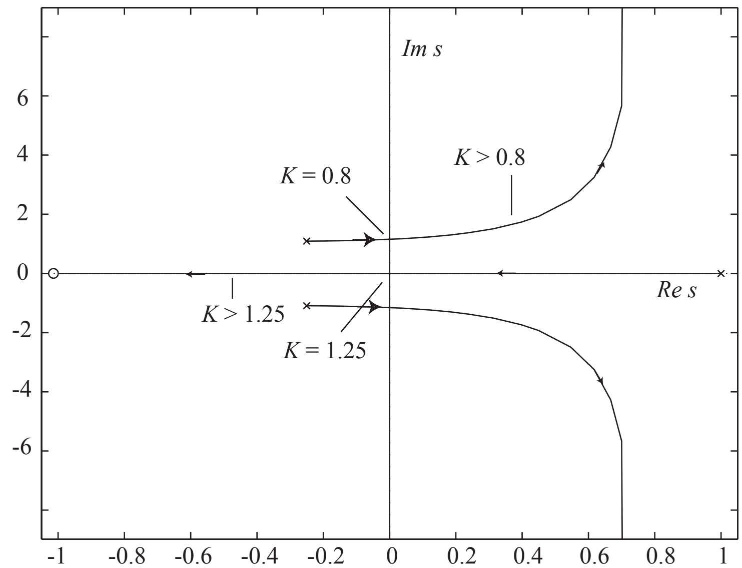

Figure 14.10: Root locus for example 14.20. Stability requires $K > 1.25$ and $K < 0.6$. Therefore, there is no stabilizing proportional compensator.

The first term is the transient response, y_{tr}, and the second term is the steady-state response $y_{ss} = R_0 u$. The transient response goes to zero, since from (14.6), $Re\{p_i(K)\} < 0$ for all i. The input is the unit step, $x = u$. So the steady-state tracking error is $|R_0 - 1|$, which depends on R_0. It is easy to find the residue R_0. We simply multiply both sides of (14.7) by s and evaluate both sides at $s = 0$, to get

$$R_0 = \hat{G}(0) = \frac{K\hat{H}_2(0)}{1 + K\hat{H}_2(0)} .$$

To have zero steady-state error, we want $R_0 = 1$, which can only happen if $\hat{H}_2(0) = \infty$. But this means $s = 0$ must be a pole of the plant transfer function $\hat{H}_2$. (This is the case in the examples of the helicopter and the DC motor.) If the plant does not have a pole at $s = 0$, the steady-state error will be

$$\left| 1 - \frac{K\hat{H}_2(0)}{1 + K\hat{H}_2(0)} \right| .$$

This error is smaller the larger the gain K. So to minimize the steady-state error we should choose as large a gain as possible, subject to the stability requirement (14.6).

However, a large value of K may lead to poor transient behavior by causing overshoots, as happened in the DC motor example in figure 14.9 for the larger gain $K = 0.0025$. To decide the appropriate K is a matter of trial and error. One studies the transient response for different (stabilizing) values of K (as we did for the DC motor) and selects K that gives a satisfactory transient behavior.

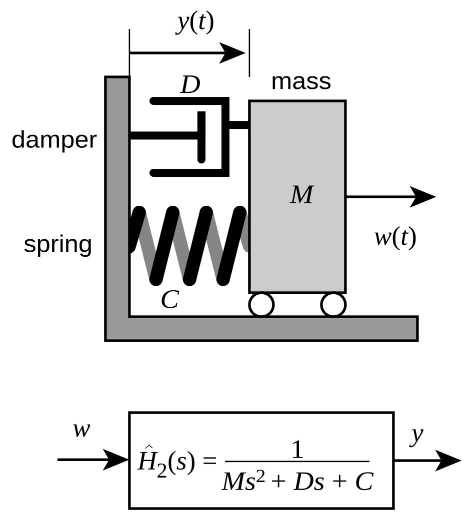

Figure 14.11: A mass-spring-damper system.

14.4 PID controllers

The P controller discussed in the previous section achieves zero steady-state error if the plant has a pole at $s = 0$. This means that the plant includes an integrator, since the transfer function of an integrator is $1/s$, which has a pole at $s = 0$. If the plant does not have a pole at $s = 0$, however, a non-zero error results. While this error can be reduced by choosing a large gain K in the controller, this results in poor transient behavior.

In this section, we develop the well-known **PID controller**, which includes an integrator in the controller. It can achieve zero steady-state error even if the plant does not have a pole at $s = 0$, and still achieve reasonable transient behavior. The PID controller is a generalization of the P controller, in that with certain choices of parameters, it becomes a P controller.

We begin with an example that has rich enough dynamics to demonstrate the strengths of the PID controller. This example describes a mechanical system, but just about any physical system that is modeled by a linear second-order differential equation is subject to similar analysis. This includes, for example, electrical circuits having resistors, capcitors, and inductors.

Example 14.21: A basic **mass-spring-damper system** is illustrated in figure 14.11. This system has a mass M that slides on a frictionless surface, a spring that attaches the mass to a fixed physical object, and a damper, which absorbes mechanical energy. A damper might be, for example, a dashpot, which is a cylinder filled with oil plus a piston. A familiar example of such a damper is a shock absorber in the suspension system of a car.

Suppose that an external force w is applied to the mass, where w is a continuous-time signal. The differential equation governing the system is obtained by setting the sum of all forces to zero,

$$M\ddot{y}(t) + D\dot{y}(t) + Cy(t) = w(t).$$

The output $y(t)$ is the position of the mass at time t, $M\ddot{y}(t)$ is the inertial force, $D\dot{y}(t)$ is the damping force due to the damper, $Cy(t)$ is the restoring force of the spring, and $w(t)$ is the externally applied force. We assume that $y(t) = 0$ when the spring is in its equilibrium position (neither extended nor compressed). M, D, and C are constants. Taking the Laplace transform, using the differentiation property from table13.4, gives

$$s^2\hat{Y}(s) + Ds\hat{Y}(s) + CY(s) = W(s),$$

so the plant or open loop transfer function is

$$\hat{H}_2(s) = \frac{\hat{Y}(s)}{\hat{W}(s)} = \frac{1}{Ms^2 + Ds + C}\ .$$

Suppose for example that the constants have values $M = 1$, $D = 1$,and $C = 1.25$. Then

$$\hat{H}_2(s) = \frac{1}{s^2 + s + 1.25}\ . \tag{14.8}$$

In this case, the transfer function has a pair of complex poles at $s = -0.5 \pm i$. Since their real part is strictly negative, the system is stable.

Suppose we wish to drive the system to move the mass to the right one unit of distance at time $t = 0$. We can apply an input force that is a unit step, scaled so that the steady-state response places the mass at position $y(t) = 1$. The final steady-state output is determined by the DC gain, which is $\hat{H}_2(0) = 1/1.25 = 0.8$, so we can apply an input given by

$$\forall\, t, \quad , w(t) = \frac{1}{0.8}u(t) = 1.25u(t),$$

where u is the unit step signal. The resulting response y_O has Laplace transform

$$\hat{Y}_O(s) = \frac{1}{s^2 + s + 1.25} \cdot \frac{1.25}{s} = \frac{-0.5 + 0.25i}{s + 0.5 - i} + \frac{-0.5 - 0.25i}{s + 0.5 + i} + \frac{1}{s}\ .$$

We call this the open-loop step response, because there is no control loop (yet).

Taking the inverse transform gives the open-loop step response

$$\forall\, t, \quad y_O(t) = e^{-0.5t}[(-0.5 + 0.25i)e^{it} + (-0.5 - 0.25i)e^{-it}]u(t) + u(t).$$

By combining the complex conjugate terms, this can be expressed as

$$\forall t, \quad y_O(t) = Re^{-0.5t}\cos(t + \theta)u(t) + u(t),$$

where $R = 1.12$ and $\theta = 2.68$. Figure 14.12 displays a plot of this open-loop step response y_O. Notice that the mass settles to position $y(t) = 1$ for large t.

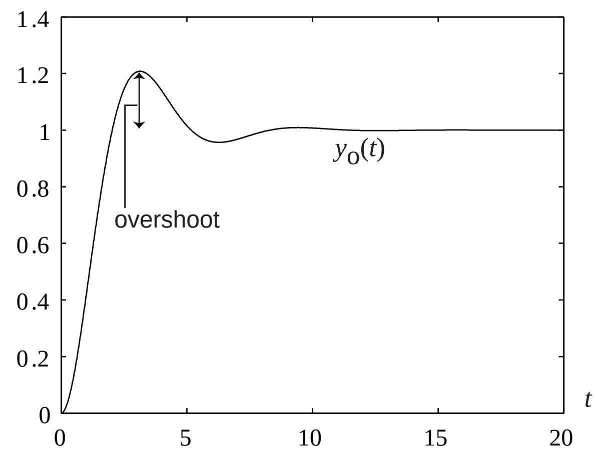

Figure 14.12: The open loop step response y_O of the mass-spring-damper system.

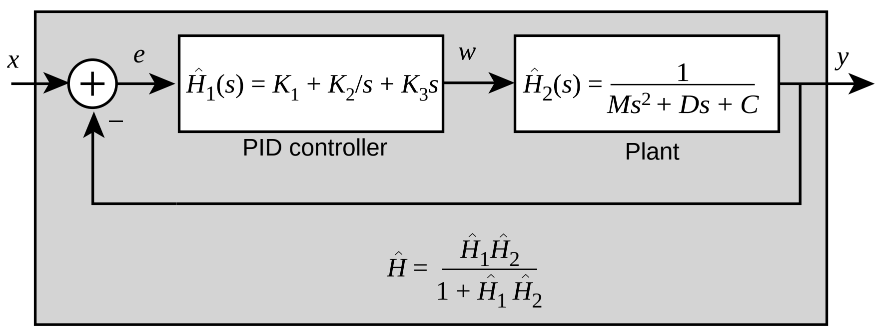

Figure 14.13: The mass-spring-damper system composed with a PID controller in a feedback composition.

This system in the previous example is stable, and therefore does not need a feedback control loop to stabilize it. However, there are two difficulties with its open-loop response, shown in figure 14.12, that can be corrected using a controller. First, it takes approximately 10 units of time for the transient to disappear, which may be too slow for some applications. Moreover, there is an overshoot of 20 percent beyond the final steady-state value, which may be too much.

We can correct for the slow response and the large overshoot, using a **PID controller**. The term 'PID' stands for proportional plus integral plus derivative. A PID controller generalizes the P controller of the previous section by adding an integral and derivative term.

The general form of the transfer function of a PID controller is

$$\hat{H}_1(s) = K_1 + \frac{K_2}{s} + K_3 s = \frac{K_3 s^2 + K_1 s + K_2}{s} . \tag{14.9}$$

We will compose this with the plant in a feedback loop, as shown in figure 14.13. Here K_1, K_2, K_3 are constants to be selected by the designer. If $K_2 = K_3 = 0$, then we have a P controller. If $K_1 = K_3 = 0$, $\hat{H}_1(s) = K_2/s$, we have an **integral contoller**, so called because $1/s$ is the transfer function of an integrator. That is, if the input to the integral controller is e, and the output is w, then

$$\forall\, t, \quad w(t) = K_2 \int_{-\infty}^{t} e(\tau) d\tau.$$

If $K_1 = K_2 = 0$, $\hat{H}_2(s) = K_3 s$, then we have a **derivative controller**, so called because s is the transfer function of a differentiator. That is, if the input to the derivative controller is e, and the output is w, then

$$\forall\, t, \quad w(t) = K_3 \dot{e}(t).$$

The following table offers guidelines for selecting the parameters of a PID controller. Of course, these are guidelines only—the actual performance of the closed loop system depends on the plant transfer function and must be checked in detail.

Parameter	**Response speed**	**Overshoot**	**Steady-state error**
K_1	Faster	Larger	Decreases
K_2	Faster	Larger	Zero
K_3	Minor change	Smaller	Minor change

Example 14.22: We now evaluate a PID controller for the mass-spring-damper system of figure 14.11, using the feedback composition of figure 14.6. We assume the parameters values $M = 1$, $D = 1$, and $C = 1.25$, as in example 14.21. The closed-loop transfer function with the PID controller is

$$\hat{H}(s) = \frac{\hat{H}_1(s)\hat{H}_2(s)}{1 + \hat{H}_1(s)\hat{H}_2(s)} = \frac{K_3 s^2 + K_1 s + K_2}{s^3 + (1 + K_3)s^2 + (1.25 + K_1)s + K_2} .$$

Suppose we provide as input a unit step. This means that we wish to move the mass to the right one unit of distance, starting at time $t = 0$. The controller will attempt to track this input. The response to a unit step input has Laplace transform

$$\hat{Y}_{pid}(s) = \hat{H}(s) \cdot \frac{1}{s} = \frac{K_3 s^2 + K_1 s + K_2}{s^3 + (1 + K_3)s^2 + (1.25 + K_1)s + K_2} \cdot \frac{1}{s} . \tag{14.10}$$

We now need to select the values for the parameters of the PID controller, K_1, K_2, and K_3. We first try proportional control with $K_1 = 10$, and $K_2 = K_3 = 0$. In this case, the step response has the Laplace transform

$$\hat{Y}_p(s) = \frac{10}{s^2 + s + 11.25} \cdot \frac{1}{s} .$$

The inverse Laplace transform gives the time response y_p, which is plotted in figure 14.14. The steady-state value is determined by the DC gain of the closed loop transfer function,

$$\left. \frac{10}{s^2 + s + 11.25} \right|_{s=0} = \frac{10}{11.25} \approx 0.89.$$

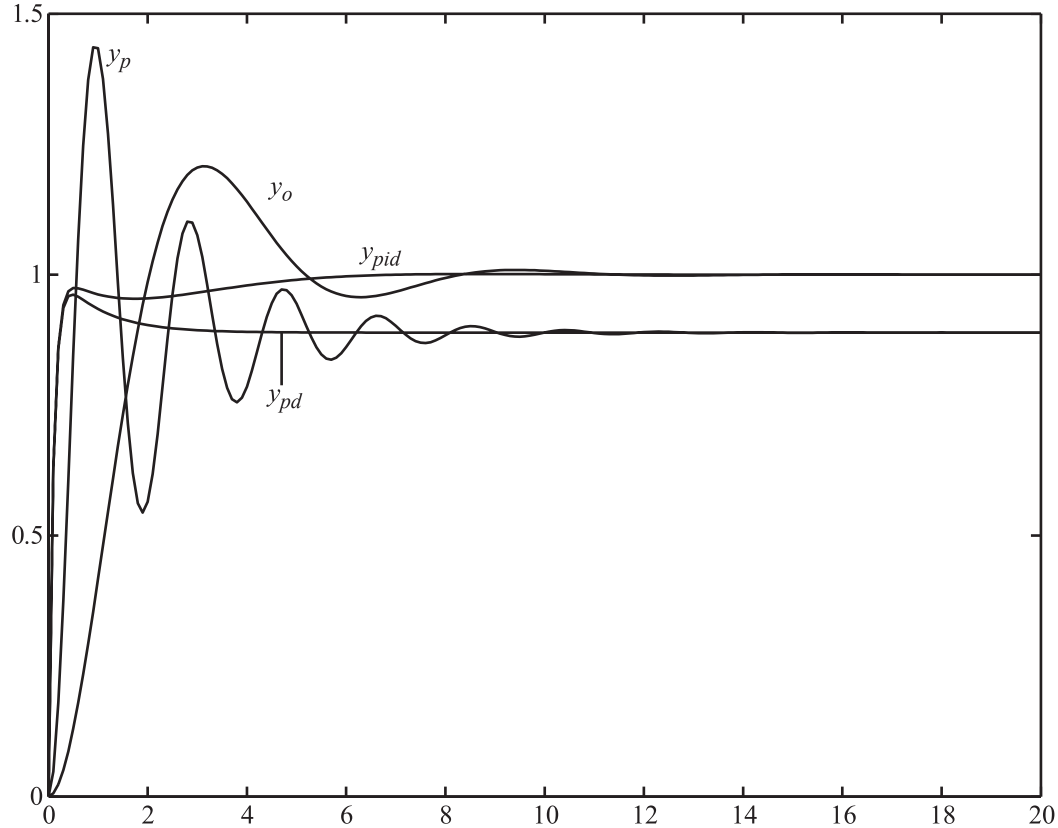

Figure 14.14: The step response for open loop, y_o, with P-control, y_p, PD-control, y_{pd}, and PID-control, y_{pid}.

This yields an error of 11 percent, and the overshoot of 50 percent is much worse than that of the open-loop response y_o, also shown in the figure. Thus, a P controller with gain $K = 10$ is useless for this application.

Following the guidelines in the table above, we add derivative control to reduce the overshoot. The result is a so-called **PD controller**, because it adds a proportional and a derivative term. For the PD controller we choose $K_1 = 10$ and $K_3 = 10$. Substitution in (14.10) gives the Laplace transform of the step response,

$$\hat{Y}_{pd}(s) = \frac{10s+10}{s^2+11s+11.25} \cdot \frac{1}{s} \, .$$

The steady-state value is given by the DC gain of the closed loop transfer function,

$$\left. \frac{10s+10}{s^2+11s+11.25} \right|_{s=0} \approx 0.89,$$

which is the same as the steady-state value for the P controller. The inverse Laplace transform gives the time response y_{pd}, which is plotted in figure 14.14. The overshoot is reduced to 10 percent—a large improvement. Also, the response is quicker—the transient disappears in about 4 time units.

Finally, to eliminate the steady-state error we add integral control. For the PID controller we choose $K_1 = 10, K_2 = 5, K_3 = 10$. Substitution in (14.10) gives the Laplace transform of the step response

$$\hat{Y}_{pid}(s) = \frac{10s^2+10s+5}{s^3+11s^2+11.25s+5} \cdot \frac{1}{s} \, .$$

The steady-state value is again given by the DC gain of the closed loop transfer function,

$$\left. \frac{10s^2+10s+5}{s^3+11s^2+11.25s+5} \right|_{s=0} = 1.$$

So the steady-state error is eliminated, as expected. The time response y_{pid} is plotted in figure 14.14. It shows significant improvement over the other responses. There is no overshoot, and the transient disappears in about 4 time units. Further tuning of the parameters K_1, K_2, K_3 could yield small improvements.

14.5 Summary

This chapter considers cascade, parallel, and feedback compositions of LTI systems described by Z or Laplace transforms. Cascade composition is applied to equalization, parallel composition is applied to noise cancellation, and feedback composition is applied to control.

Because we are using Z and Laplace transforms rather than Fourier transforms, we are able to consider unstable systems. In particular, we find that while, in principle, cascade and parallel compositions can be used to stabilize unstable systems, the result is not robust. Small changes in parameter values can result in the system being once again unstable. Feedback composition, on the other hand, can be used to robustly stabilize unstable systems. We illustrate this first with a simple helicopter example. The second example, a DC motor, benefits from more sophisticated controllers. The third example, a mass-spring-damper system, motivates the development of the well-known PID controller structure. PID controllers can be used to stabilize unstable systems and to improve the response time, precision, and overshoot of stable systems.

Exercises

Each problem is annotated with the letter **E, T, C** which stands for exercise, requires some thought, requires some conceptualization. Problems labeled **E** are usually mechanical, those labeled **T** require a plan of attack, those labeled **C** usually have more than one defensible answer.

1. **E** This exercise studies equalization when the channel is only known approximately. Consider the cascade composition of figure 14.1, where $\hat{H}_1$ is the channel to be equalized, and $\hat{H}_2$ is the equalizer. If the equalizer is working perfectly, then $x = y$. For example, if

$$\hat{H}_1(z) = \frac{z}{z-0.5} \quad \text{and} \quad \hat{H}_2(z) = \frac{z-0.5}{z} ,$$

then $x = y$ because $\hat{H}_1(z)\hat{H}_2(z) = 1$.

   (a) Suppose that $\hat{H}_2(z)$ is as given above, but the plant is a bit different,

$$\hat{H}_1(z) = \frac{z}{z-0.5-\varepsilon} .$$

   Suppose that $x = \delta$, the Kronecker delta function. Plot $y - x$ for $\varepsilon = 0.1$ and $\varepsilon = -0.1$.

   (b) Now suppose that the equalizer is

$$\hat{H}_2(z) = \frac{z-2}{z} ,$$

   and the channel is

$$\hat{H}_1(z) = \frac{z}{z-2-\varepsilon} .$$

   Again suppose that $x = \delta$, the Kronecker delta function. Plot $y - x$ for $\varepsilon = 0.1, -0.1$.

   (c) For part (b), show that equalization error $y - x$ grows without bound whenever $\varepsilon \neq 0$, $|\varepsilon| < 1$.

2. **E** This exercise studies equalization for continuous-time channels. Consider the cascade composition of figure 14.1, where $\hat{H}_1$ is the channel to be equalized, and $\hat{H}_2$ is the equalizer. Both are causal. If

$$\hat{H}_1(s) = \frac{s+1}{s+2} \quad \text{and} \quad \hat{H}_2(s) = \frac{s+2}{s+1} ,$$

then $x = y$ because $\hat{H}_1(s)\hat{H}_2(s) = 1$.

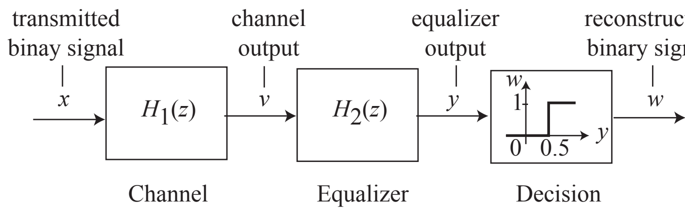

Figure 14.15: Arrangement of decision-directed equalization of exercise 3.

(a) Suppose $\hat{H}_2$ is as above but

$$\hat{H}_1(s) = \frac{s+1}{s+2+\varepsilon} .$$

Suppose $x = u$, the unit step. Plot $y - x$ for $\varepsilon = 0.1$ and $\varepsilon = -0.1$, and calculate the steady state error.

(b) Now suppose the equalizer is

$$\hat{H}_2(s) = \frac{s-1}{s+2} ,$$

and the channel is

$$\hat{H}_1(s) = \frac{s+2}{s-1-\varepsilon} .$$

Again suppose that $x = u$. Plot $y - x$ for $\varepsilon = 0.1, -0.1$.

(c) For part (b) show that the error $y - x$ grows without bound for any $\varepsilon \neq 0$, $|\varepsilon| < 1$.

3. **T** This exercise explores decision-directed equalization. The arrangement is shown in figure 14.15. The transmitted signal is a binary sequence $x : Integers \to \{0,1\}$. The causal channel transfer function is $\hat{H}_1$ and the equalizer transfer function is $\hat{H}_2$. The channel output is the real-valued signal $v : Integers \to Reals$. The equalizer output is the real-valued signal $y : Integers \to Reals$. This signal is fed to a decision unit whose binary output at time n, $w(n) = 0$ if $y(n) < 0.5$ and $w(n) = 1$ if $y(n) \geq 0.5$. Thus the decision unit is a (nonlinear) memoryless system,

$$Decision : [Integers \to Reals] \to [Integers \to \{0,1\}],$$

defined by a threshold rule

$$\forall n, \quad (Decision(y))(n) = \begin{cases} 0, & y(n) < 0.5, \\ 1, & y(n) \geq 0.5 \end{cases}$$

At each point in time, the receiver has an estimate $\hat{H}_1^e$ of the true channel transfer function, $\hat{H}_1$. The equalizer is set at

$$\hat{H}_2(z) = [\hat{H}_1^e(z)]^{-1}. \tag{14.11}$$

(a) Suppose that initially $\hat{H}_1(z) = \frac{z}{z-0.2}$, and the estimate is perfect, $\hat{H}_1^e = \hat{H}_1$. (This perfect estimate is achieved using a known training sequence for x.) Determine the respective impulse responses h_1 and h_2.

Now suppose the signal x is

$$\forall n, \quad x(n) = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0, n \text{ even} \\ 0, & n \geq 0, n \text{ odd} \end{cases} \tag{14.12}$$

Calculate the channel output $v(n) = (h_1 * x)(n), n \leq 3$. Then calculate the equalizer output $y(n) = (h_2 * v)(n), n \leq 3$, and check that $y(n) = x(n), n \leq 3$. Also check that $w(n) = x(n), n \leq 3$.

(b) Now suppose the channel transfer function has changed to

$$\hat{H}_1(z) = \frac{z}{z - 0.3},$$

but the receiver's estimate hasn't changed, i.e.

$$\hat{H}_1^e(z) = \frac{z}{z - 0.2},$$

so the equalizer (14.11) hasn't change either. For the same input signal again calculate the channel and equalizer outputs $v(n), y(n), n \leq 3$. Check that $y(n) \neq x(n), n > 0$. But show that the decision $w(n) = x(n), n \leq 3$. So the equalizer correctly determines x.

(c) Since the receiver's decision $w = x$, it can make a new estimate of the channel using the fact that $\hat{Y} = \hat{H}_1 \hat{H}_2 \hat{X} = \hat{H}_1 \hat{H}_2 \hat{W}$. The new estimate is

$$\hat{H}_1^e = \frac{\hat{Y}}{\hat{H}_2 \hat{W}} \; . \tag{14.13}$$

Suppose time is 3, and the receiver has observed $y(n), w(n), n \leq 3$. Since the Z transforms $\hat{Y}$ and $\hat{W}$ also depend on values of $y(n), w(n)$ for $n > 3$, these Z transforms can *not* be calculated at time $n = 3$, and so the estimator (14.13) cannot be used. The following approach will work, however.

Suppose the receiver knows that the unknown channel transfer function is of the form

$$\hat{H}_1(z) = \frac{z}{z - a} \; ,$$

so that only the parameter a has to be estimated. Using this information, we have

$$\hat{Y}(z) = \frac{z}{z - a} \, \frac{z - 0.2}{z} \, \hat{W}(z) = \frac{z - 0.2}{z - a} \, W(z).$$

Now take the inverse Z transform and express the time-domain relation between y and w. Show that you can estimate a knowing $y(0), y(1), w(0), w(1)$.

4. **E** This continues exercise 3. It shows that if the channel estimate $\hat{H}_1^e$ is not sufficiently close to the true channel $\hat{H}_1$, the decision may become incorrect. Suppose the true channel is $\hat{H}_1(z) = \frac{z}{z-a}$, the estimate is $\hat{H}_1^e(z) = \frac{z}{z-0.2}$, the equalizer is $\hat{H}_2(z) = [\hat{H}_1^e(z)]^{-1} = \frac{z-0.2}{z}$, and the decision is as in figure 14.15. Assume the input signal x to be the same as in (14.12) Show that if $a = 0.6$ then $w(0) = x(0), w(1) = x(1), w(2) = x(2)$, but $w(3) \neq x(3)$.

5. **E** This exercise continues the discussion in examples 14.5, 14.6 for the continuous-time, causal and stable channel with impulse response h_1 and transfer function

$$\forall s \in RoC(h_1) = \{s \mid Re\{s\} > -1\}, \quad \hat{H}_1(s) = \frac{s-2}{s+1} .$$

   (a) Calculate h_1 and sketch it. (Observe how the zero in the right-half plane at $s = 2$ accounts for the negative values.)

   (b) The inverse of $\hat{H}_1$,

$$\hat{H}_2(s) = \frac{s+1}{s-2} ,$$

   has a pole at $s = 2$. So as a causal system, the inverse is unstable. But as a non-causal system, it is stable with $RoC = \{s \mid Re\{s\} < 2\}$ which includes the imaginary axis. Evaluate the impulse response h_2 of $\hat{H}_2$ as an anti-causal system, and give a sketch.

   (c) The impulse response h_2 calculated in (a) is non-zero for $t \leq 0$. Consider the finite-duration, anti-causal impulse response h_3 obtained by truncating h_2 before time -5,

$$\forall t \in Reals, \quad h_3(t) = \begin{cases} h_2(t), & t \geq -5 \\ 0, & t < -5 \end{cases}$$

   and sketch h_3. Calculate the transfer function $\hat{H}_3$, including its RoC, by using the definition of the Laplace transform.

   (d) Obtain the causal impulse response h_4 by delaying h_3 by time T, i.e.

$$\forall t \in Reals, \quad h_4(t) = h_3(t+T).$$

   Sketch h_4 and find its transfer function, $\hat{H}_4$. Then $\hat{H}_4$ is an approximate inverse of $\hat{H}_1$ with a delay of 5 time units. (**Note**: h_3 has a delta function at 0.)

6. **T** The proportional controller of figure 14.7 stabilizes the plant for $K > 1$. In this exercise, we try to achieve the same effect by the cascade compensator of figure 14.1.

   (a) Assume that the plant $\hat{H}_2$ is as given in figure 14.7. Design $\hat{H}_1$ for the cascade composition of figure 14.1 so that $\hat{H}_2\hat{H}_1$ is the same as the closed-loop transfer function achieved in figure 14.7.

   (b) Now suppose that the model of the plant is not perfect, and the plant's real transfer function is

$$\hat{H}_2(s) = \frac{1}{M(s-\varepsilon)},$$

   for some small value of $\varepsilon > 0$. Using the same $\hat{H}_1$ that you designed in part (a), what is the transfer function of the cascade composition? Is it stable?

7. **T** Consider a discrete-time causal plant with transfer function

$$\hat{H}_2(z) = \frac{z}{z-2} .$$

   (a) Where are the poles and zeros? Is the plant stable?

(b) Find the impulse response of the plant. Is it bounded?

(c) Give the closed-loop transfer function for the P controller for this plant.

(d) Sketch the root locus for the P controller for this plant.

(e) For what values of K is the closed-loop system stable?

(f) Find the step response of the closed-loop system. Identify the transient and steady-state responses. For $K = 10$, what is the steady-state tracking error?

(g) Suppose that the plant is instead given by

$$\hat{H}_2 = \frac{z}{z - 2 - \varepsilon},$$

for some real $\varepsilon \geq 0$. For what values of K is the P controller robust for plants with $|\varepsilon| < 0.5$?

8. **T** Consider the feedback composition in figure 14.6. Suppose that $\hat{H}_1\hat{H}_2$ is causal and **strictly proper**, meaning that the order of the numerator is greater than the order of the denominator.

(a) Show that if $\hat{H}_1\hat{H}_2$ is causal and strictly proper, then so is $\hat{H}$, the transfer function of the feedback composition given by (14.1).

(b) For the discrete-time case, show that we can write

$$\hat{H}_1(z)\hat{H}_2(z) = z^{-1}\hat{G}(z), \tag{14.14}$$

where $G(z)$ is proper, and is the transfer function of a causal system. Intuitively, this means that there must be a net unit delay in the feedback loop, because z^{-1} is the transfer function of a unit delay.

(c) Use the result of part (a) to argue that the system $\hat{H}_1\hat{H}_2$ has state-determined output.

(d) For the continuous-time case, show that we can write

$$\hat{H}_1(s)\hat{H}_2(s) = s^{-1}\hat{G}(s), \tag{14.15}$$

where $G(s)$ is proper, and is the transfer function of a causal system. Intuitively, this means that there must be an integration in the feedback loop, because s^{-1} is the transfer function of an integrator.

(e) Use the result of part (c) to argue that the system $\hat{H}_1\hat{H}_2$ has state-determined output, assuming that the input is bounded and piecewise continuous.

9. **E** Consider the mth order polynomial $s^N + a_{m-1}s^{m-1} + \cdots + a_1 s + a_0$. Suppose all its roots have negative real parts. Show that all coefficients of the polynomial must be positive, i.e., $a_{m-1} > 0, \cdots, a_0 > 0$. Hint. Express the polynomial as $(s - p_1)\cdots(s - p_m)$ with $Re\{p_i\} > 0$. Note that complex roots must occur in complex conjugate pairs. (The positiveness of all coefficients is a necessary condition. A sufficient condition is given by the **Routh-Hurwitz** criterion, described in control theory texts.)

10. **T** Consider the feedback composition in figure 14.6. The plant's transfer function is $\hat{H}_2(s) = 1/s^2$.

(a) Show that no PI controller in the form $\hat{H}_1(s) = K_1 + K_2/s$ can stabilize the closed loop system for any values of K_1, K_2. Hint. Use the result of problem 9.

(b) Show that by the proper choice of the coefficients K_1, K_2 of a PD controller in the form $\hat{H}_1(s) = K_1 + K_2 s$, you can place the closed-loop poles at any locations p_1, p_2 (these must be complex conjugate if they are complex).

11. **T** Consider the feedback composition in figure 14.6. The plant's transfer function is $\hat{H}_2(s) = 1/(s^2 + 2s + 1)$. The PI controller is $\hat{H}_1(s) = K_1 + K_2/s$.

    (a) Take $K_2 = 0$, and plot the root locus as K_1 varies. For what values of K_1 is the closed loop system stable? What is the steady state error to a step input as a function of K_1?

    (b) Select K_1, K_2 such that the closed loop system is stable and has zero-steady state error.