HETEROGENEOUS AD HOC NETWORKS

Béla Bollobás
Memphis and Cambridge
bollobas@dpmms.cam.ac.uk

with

Svante Janson (Uppsala)   Oliver Riordan (Cambridge)

ITR Review
October 4, 2006
Most large-scale real-life networks are far from homogeneous: vertices may have been ‘born’ at different times, with old and new vertices having very different properties, or the locations may influence their roles, etc. Many examples: sensor networks, WWW, protein interaction networks, telephone call graphs, scientific collaboration graphs, protein domains, models of the brain, food webs, gene regulation, traffic networks, economic networks, etc.

The spread of degrees is often very large. In particular, in many examples the degree distribution follows a power-law.

[Contrary to this, the models in the classical theory, started by Erdős and Rényi, namely $G(n, p)$, $G(n, M)$, $G_{k-\text{reg}}$, $G_{k-\text{out}}$, etc., are all homogeneous.]
Sparse Random Graphs

Almost all large-scale real-life networks are sparse: the average degree remains bounded as the order increases.

Questions concerning sparse random networks:

Is there a giant (linear-size) component? How fast does it emerge? Order of phase transition?

How large is the second largest component?

Is the system robust? Is it vulnerable?

What is the diameter of the giant component?

What is the distribution of the cycles?
The Classical Phase Transition

**Erdős and Rényi:** \( G_n = G(n, c/n) \):

- \( n \) labelled (distinguishable) vertices,
- each edge is present with probability \( c/n \).

If \( c < 1 \) then whp

\[ |C_1(G_n)| = O(\log n), \]

If \( c > 1 \) then whp

\[ |C_1(G_n)| \sim \theta(c)n. \]
Take a function $\kappa(x, y)$,
$x, y \in V(G) = \{1, 2, \ldots, n\}$. Join $x$ to $y$ with probability $c\kappa(x, y)/n$. 

[Note the factors $c$ and $1/n$.]

What is the critical factor $c = c_0$?
A Simple Form of General Question

In the ER case: $\kappa(x, y) \equiv 1$ and the critical factor is $c_0 = 1$.
But the known results do not answer the simplest inhomogeneous variants:

$$V = V_1 \cup \cdots \cup V_k, \quad |V_i| = n_i,$$
$$\kappa(x, y) = \kappa_{ij} \text{ if } x \in V_i, \ y \in V_j.$$

What is the critical constant for this inhomogeneous graph?

Thus, if $x$ is joined to $y$ with probability $c\kappa_{ij}/n$, for what $c$ is there a giant component?
A Toy Example

Let \( p, r, s > 0 \).

\[
\frac{n}{2} \quad \frac{cp}{n} \quad \frac{cs}{n} \quad \frac{cr}{n} \quad \frac{n}{2}
\]

What is the critical factor \( c_0 \)?
A Toy Example

Example: \( p = 4, r = 1, s = 2 \).

What is the critical factor \( c_0 \)?
Recently, Svante Janson (Uppsala), Oliver Riordan (Cambridge) and I have given a very general model for inhomogeneous random graphs, a ‘meta-model’. This contains (exactly) many earlier models.

We have established a close connection between the component structure of a random graph in this model, the survival probability of a related branching process, and the norm of a certain integral operator.

Here I’ll give a greatly simplified account of (a small fraction of) the results.
A vertex space $\mathcal{V}$ is a triple $(\mathcal{S}, \mu, (x_n)_{n \geq 1})$ s.t.

(i) $\mathcal{S}$ is a separable metric space;
(ii) $\mu$ is a (positive) Borel measure on $\mathcal{S}$ with $0 < \mu(\mathcal{S}) < \infty$;
(iii) for each $n$ (or just infinitely many $n$), $x_n$ is a random sequence $(x_1, \ldots, x_{N_n})$ of $N_n$ points of $\mathcal{S}$.
(iv) $\mu_n \rightarrow \mu$ in $M(\mathcal{S})$: $\mu_n$ converges to $\mu$ in probability.

A kernel is a symmetric, non-negative, Borel-measurable function $\kappa$ on $\mathcal{S} \times \mathcal{S}$. 
The random graph $G^\mathcal{V}(n, \kappa)$

The random graph with vertex space $\mathcal{V}$ and kernel $\kappa$, $G^\mathcal{V}(n, \kappa)$, is defined as follows.

The vertex set is $\{1, \ldots, N_n\}$, and for $i \neq j$, join $i$ to $j$ with probability

$$p_{ij} = \min\left\{ \frac{\kappa(x_i, x_j)}{n}, 1 \right\},$$

with all choices independent.

We assume that $\kappa$ is graphical: (v) $\kappa$ is continuous a.e. on $S \times S$; (vi) $\kappa \in L^1(S \times S, \mu \times \mu)$; and (vii) the expectation $\mathbb{E}_{e}(G^\mathcal{V}(n, \kappa)) / n$ tends to

$$\frac{1}{2} \iint_{S^2} \kappa(x, y) d\mu(x) d\mu(y).$$
Branching Processes

To study the components of $G(n, \kappa)$, we use the multi-type Galton–Watson branching process with type space $\mathcal{S}$, where a particle of type $x \in \mathcal{S}$ is replaced in the next generation by a set of particles distributed as a Poisson process on $\mathcal{S}$ with intensity $\kappa(x, y) \, d\mu(y)$. (Thus, the number of children with types in a subset $A \subseteq \mathcal{S}$ has a Poisson distribution with mean $\int_A \kappa(x, y) \, d\mu(y)$.)

We denote this branching process, started with a single particle of type $x$, by $X_\kappa(x)$.

We write $X_\kappa$ for the same process with the type of the initial particle chosen at random, with distribution $\mu$. 
Survival Probabilities

Let $\rho_k(\kappa; x)$ be the probability that the branching process $X_\kappa(x)$ has a total population of exactly $k$ particles, and let $\rho_{\geq k}(\kappa; x)$ be the probability that the total population is at least $k$.

$\rho(\kappa; x)$ is the probability that the branching process $X_\kappa(x)$ survives for eternity.

Similarly, $\rho(\kappa)$ is the probability that the branching process $X_\kappa$ survives for eternity.
Operators

Let $T_\kappa$ be the integral operator on $(S, \mu)$ with kernel $\kappa$, defined by

$$(T_\kappa f)(x) = \int_S \kappa(x, y) f(y) \, d\mu(y)$$

for any (measurable) function $f$ such that this integral is defined (finite or $+\infty$) for a.e. $x$.

$$\|T_\kappa\| = \sup \{ \|T_\kappa f\|_2 : f \geq 0, \|f\|_2 \leq 1 \} \leq \infty.$$
Basic Results: the Giant Component

Let \((\kappa_n)\) be a graphical sequence of kernels on a vertex space \(V\) with limit \(\kappa\).

If \(\|T_\kappa\| \leq 1\), then \(C_1(G^V(n, \kappa_n)) = o_p(n)\);

if \(\|T_\kappa\| > 1\), then \(C_1(G^V(n, \kappa_n)) = \Theta(n)\) whp.

If \(\kappa\) is irreducible, then

\[
\frac{1}{n} C_1(G^V(n, \kappa_n)) \xrightarrow{p} \rho(\kappa).
\]

In all cases \(\rho(\kappa) < 1\); furthermore, \(\rho(\kappa) > 0\) if and only if \(\|T_\kappa\| > 1\).
Solving Our Toy Example

The vertex space is \((0, 1]\) with the Lebesgue measure, and the kernel is

\[
\kappa(x, y) = \begin{cases} 
p & \text{if } x, y \leq 1/2, \\
r & \text{if } x, y > 1/2, \\
s & \text{otherwise.}
\end{cases}
\]

To find the critical factor \(c_0\), we have to determine the operator norm (i.e., the largest eigenvalue) \(\lambda_1\) of the corresponding \(2 \times 2\) matrix: \(c_0 = 2/\lambda_1\).
When $p = 4$, $r = 1$ and $s = 2$, we find that $c_0 = 2/5$, so the critical system is this:
The rank 1 case

A very natural special case: it includes many random graph models that have been widely studied. The kernel $\kappa$ has the form $\kappa(x, y) = \psi(x)\psi(y)$ for some function $\psi > 0$ on $S$. (We assume $\int \psi \, d\mu < \infty$, but not necessarily that $\int \psi^2 \, d\mu < \infty$.)

The function $\psi(x)$ can be interpreted as the “activity” of a vertex at $x$, with the probability of an edge between two vertices proportional to the product of their activities. In the rank 1 case, $T_\kappa f = (\int f \psi) \psi$, so

$$\|T_\kappa\| = \|\psi\|_2^2 = \int \psi^2 \, d\mu \leq \infty.$$
The rank 1 case

The operator $T_\kappa: f \rightarrow \left( \int f \psi \right) \psi$ is bounded if and only if $\psi \in L^2$, in which case $T_\kappa$ has rank 1, so it is compact, and $\psi$ is the unique eigenfunction with maximal eigenvalue.

The distribution of vertex degrees is governed by the distribution of the function $\lambda(x) = (\int \psi d\mu)\psi(x)$ on $(\mathcal{S}, \mu)$. In particular, the degree sequence will (asymptotically) have a power-law tail if the distribution of $\lambda(x)$ has; for example, if $\mathcal{S} = (0, 1]$ with $\mu$ Lebesgue measure, and $\psi(x) = cx^{-1/p}$.

Special cases of this model have been studied by many people, including Chung and Lu, Norros and Reittu, Britton, Deijfen and Martin-Löf, Luczak, Molloy and Reed, and Aiello, Chung and Lu, Van der Hofstad, Hooghiemstra and Van Mieghem, and Fernholz and Ramachandran.
Consider the kernel \( c_\kappa(x, y) = c\psi(x)\psi(y) \), with \( c > 0 \) a parameter. The threshold for \( c \) is \( c_0 = ||T_\kappa||^{-1} = (\int_S \psi^2)^{-1} \). Set

\[
\beta(t) = \int_S \left(1 - e^{-t\psi(x)}\right) \psi(x) \, d\mu(x), \quad t \geq 0,
\]

and for \( c > c_0 \) put

\[
\alpha(c) = c \int_S \rho c_\kappa \psi \, d\mu = c \beta(\alpha(c)),
\]

so that \( \alpha \) is the inverse of \( t \to t/\beta(t) \).
The rank 1 case – Results

Then the survival probability function is

$$\rho_{cK} = 1 - e^{\alpha(c)\psi},$$

and

$$\rho(cK) = \int_S \rho_{cK} d\mu(x)$$

is the asymptotic proportion of the giant component.
The case $\psi(x) = x^{-1/p}$

Take $S = (0, 1]$ with the Lebesgue measure and graphical kernel $\psi(x) = x^{-1/p}$, $1 < p \leq \infty$.

What can we say about the phase transition?

Case 1: $1 < p < 2$. In this case, $\|\psi\|_2 = \infty$, so $c_0 = 0$. Also, as $\varepsilon \to 0$,

$$\rho(\varepsilon) \sim c\varepsilon^{1/(2-p)}.$$  

[N.B. The exponent $1/(p-2)$ can take any value greater than 1.]

Case 2: $p = 2$. We still have $\|\psi\|_2 = \infty$ and thus $c_0 = 0$. As $\varepsilon \to 0$,

$$\rho(\varepsilon) = e^{-(1+o(1))/2\varepsilon};$$

we have an infinite order phase transition (but starting at 0).
The case $\psi(x) = x^{-1/p}$

Case 3: $2 < p < 3$. For $p > 2$ we have $\int \psi^2 \, d\mu < \infty$, and thus $c_0 = 1 - 2/p > 0$, so we have a genuine phase transition. Also,

$$\rho(c_0 + \varepsilon) \sim c \varepsilon^{1/(p-2)} \quad \text{as } \varepsilon \to 0.$$  

[N.B. The exponent $1/(p - 2)$ can take any value greater than 1.]

Case 4: $p = 3$. Here, $c_0 = 1/3$ and

$$\rho(c_0 + \varepsilon) \sim c\varepsilon / \ln(1/\varepsilon) \quad \text{as } \varepsilon \to 0,$$

so $\rho'(c_0) = 0$. 

HETEROGENEOUS AD HOC NETWORKS
The case $\psi(x) = x^{-1/p}$

Case 5: $3 < p \leq \infty$. In this case, $c_0 = 1 - 2/p$ and

$$\rho(c_0 + \varepsilon) \sim c\varepsilon,$$

so we have a phase transition with exponent 1.
Unsolved Problems

This is only the beginning of the road: numerous hard questions remain.

- Prove much more detailed results about the window of the phase transition.
- Construct models producing denser graphs.
- Explore the connection with the recent body of work of Borgs, Chayes, Lovász, Sós and Vesztergombi.
- What about the inverse problem? Given a sequence \((G_n)\), can we identify \(\kappa\)?