Dynamical constrained

impulse system analysis

through viability approach and applications.

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Berkeley - 21 November 2006

Outline

1. ▶ *Viability Theory and Dynamical Systems*
   - *Complex Dynamical Systems*
   - *Viability Kernels*
   - *Capture Basins*
   - *Crisis Time Function*
   - *Hybrid Systems*
   - *Exemple: Minimal time for hybrid systems, the Metropolitan*
2. *Set Valued Numerical Analysis and Control Synthesis*
3. *Some Applications*
   - *Mathematics: Fractal Sets and Attractors*
   - *Engineering : Designing discontinuous control*
   - *Embedded Systems: Lagrangian Sensors*

Part I

Viability Theory

and

Dynamical Systems

Complex Constrained Dynamical Systems

Continuous Dynamical Systems

The state of the system is represented by

- a variable $x \in K \subset X = \mathbb{R}^n$, Kcompact,
- regulated by a parameter (control) $u \in U$,
- which evolution is ruled by a continuous dynamic law :

$$\begin{array}{rcl} x'(t) & = & f(x(t), u(t)) \in X \\ u(t) & \in & \mathcal{U}(x(t)) \subset U \end{array}$$

Complex Constrained Dynamical Systems

Discrete Constrained Dynamical Systems

The state of the system is represented by

- a variable $x \in K \subset X = \mathbb{R}^n$, K compact,
- regulated by a parameter (control) $u \in U$,
- which evolution is ruled by a recursive dynamic law :

$$\boxed{\begin{array}{rcl} x^+ & = & \varphi(x^-, u) \in X \\ u & \in & \mathcal{U}(x^-) \subset U \end{array}}$$

Complex Constrained Dynamical Systems

Hybrid Dynamical Systems

The state of the system is represented by

- a variable $x \in K \subset X = \mathbb{R}^n$, K compact,
- a variable $y \in D \subset Y$ a finite set,
- regulated by a parameter (control) $u \in U$,
- reset as soon as $(x, y) \in \mathcal{R} \subset X \times Y$.
- which evolution is ruled by an hybrid dynamic law :

$$\begin{array}{lcl} (x'(t), y'(t)) & = & f(x(t), y(t), u(t)) \times \{0\} \subset X \times Y \\ (x^+(t), y^+(t)) & = & \varphi_{\mathcal{R}}(x^-(t), y^-(t), u(t)) \subset X \times Y \\ u(t) & \in & \mathcal{U}(x(t), y(t)) \subset U \end{array}$$

where $\varphi_{\mathcal{R}}(x, y, u) := \begin{cases} \varphi(x, y, u) & \text{si}(x, y) \in \mathcal{R}) \\ \emptyset & \text{si}(x, y) \notin \mathcal{R}) \end{cases}$

Complex Constrained Dynamical Systems

Hybrid Dynamical Systems with Target

The state of the system is represented by

- a variable $x \in K \subset X = \mathbb{R}^n$, K compact,
- a variable $y \in D \subset Y$ a finite set,
- regulated by a parameter (control) $u \in U$,
- reset as soon as $(x, y) \in \mathcal{R} \subset X \times Y$.
- The evolution stops as soon as a target $\mathcal{C} \subset X \times Y$ is reached in finite time,
- which evolution is ruled by an hybrid dynamic law with target:

Complex Constrained Dynamical Systems

Hybrid Dynamical Systems with Target

$$
\begin{array}{lcl}
(x'(t), y'(t)) & = & f_{\mathcal{C}}(x(t), y(t), u(t)) \times \{0\} \subset X \times Y \\
(x^+(t), y^+(t)) & = & \varphi_{\mathcal{R}}(x^-(t), y^-(t), u(t)) \subset X \times Y \\
u(t) & \in & \mathcal{U}(x(t), y(t)) \subset U
\end{array}
$$

where $f_{\mathcal{C}}(x, y, u) := \begin{cases} f(x, y, u) & \text{si}(x, y) \notin \mathcal{C}) \\ 0 & \text{si}(x, y) \in \mathcal{C}) \end{cases}$

and $\varphi_{\mathcal{R}}(x, y, u) := \begin{cases} \varphi(x, y, u) & \text{si}(x, y) \in \mathcal{R}) \\ \emptyset & \text{si}(x, y) \notin \mathcal{R}) \end{cases}$

Complex Constrained Dynamical Systems

Hybrid Tychastic Dynamical Systems with Target

The state of the system is represented by

- a variable $x \in K \subset X = \mathbb{R}^n$, K compact,
- a variable $y \in D \subset Y$ a finite set,
- regulated by a parameter (control) $u \in U$,
- and a parameter (tyche = uncertainty) $v \in V$,
- reset as soon as $(x, y) \in \mathcal{R} \subset X \times Y$.
- which evolution is ruled by an hybrid tychastic dynamic law with target:

Complex Constrained Dynamical Systems

Hybrid Tychastic Dynamical Systems with Target

$$
\begin{array}{lcl}
(x'(t), y'(t)) & = & f_{\mathcal{C}}(x(t), y(t), u(t), v(t)) \times \{0\} \subset X \times Y \\
(x^{+}(t), y^{+}(t)) & = & \varphi_{\mathcal{R}}(x^{-}(t), y^{-}(t), u(t), v(t)) \subset X \times Y \\
(u(t), v(t)) & \in & \mathcal{U}(x(t), y(t)) \times \mathcal{V}(x(t), y(t)) \subset U \times V
\end{array}
$$

where $f_{\mathcal{C}}(x, y, u, v) := \begin{cases} f(x, y, u, v) & \text{si}(x, y) \notin \mathcal{C}) \\ 0 & \text{si}(x, y) \in \mathcal{C}) \end{cases}$

and $\varphi_{\mathcal{R}}(x, y, u, v) := \begin{cases} \varphi(x, y, u, v) & \text{si}(x, y) \in \mathcal{R}) \\ \emptyset & \text{si}(x, y) \notin \mathcal{R}) \end{cases}$

Complex Constrained Dynamical Systems

General Asumptions

Let us set :

$F_{\mathcal{C}}(x,y,v) := \bigcup_{u\in U(x,y)} f_{\mathcal{C}}(x,y,u,v) \times \{0\}$

$\Phi_{\mathcal{R}}(x,y,v) := \bigcup_{u\in U(x,y)} \varphi_{\mathcal{R}}(x,y,u,v)$,

- ☞ $\forall v \in V,\ (x,y) \rightsquigarrow F_{\mathcal{C}}(x,y,v)$is closed graph, convex, compact valued and with linear growth:

  $\exists c > 0,\ \ \forall x \in X,\ \ \|F(x)\| := \sup_{y\in F(x)} \|y\| \leq c(1+\|x\|$

- ☞ $\forall v \in V,\ (x,y) \rightsquigarrow \Phi_{\mathcal{R}}(x,y,v)$is closed graph,

- ☞ K, D, $\mathcal{C}$and $\mathcal{R}$are closed,

Complex Constrained Dynamical Systems

The General Differential Inclusion:

$$
\begin{array}{lll}
(x'(t), y'(t)) & \in & F_{\mathcal{C}}(x(t), y(t), v(t)) \times \{0\} \subset X \times Y \\
(x^{+}(t), y^{+}(t)) & \in & \Phi_{\mathcal{R}}(x^{-}(t), y^{-}(t), v(t)) \subset X \times Y \\
v(t) & \in & \mathcal{V} := \{v(\cdot), measurable\ de\ [0, +\infty[\rightarrow V\}
\end{array}
$$

Viability and Capturability Concepts

Viability Theory systematically studies the properties

- of **viability of the evolutions in some environment** (maintenance of pollution thresholds, for example) at any time
- or until a finite, or prescribed, or minimal time where the evolution **reaches a given target**

For that purpose it introduces the notion of

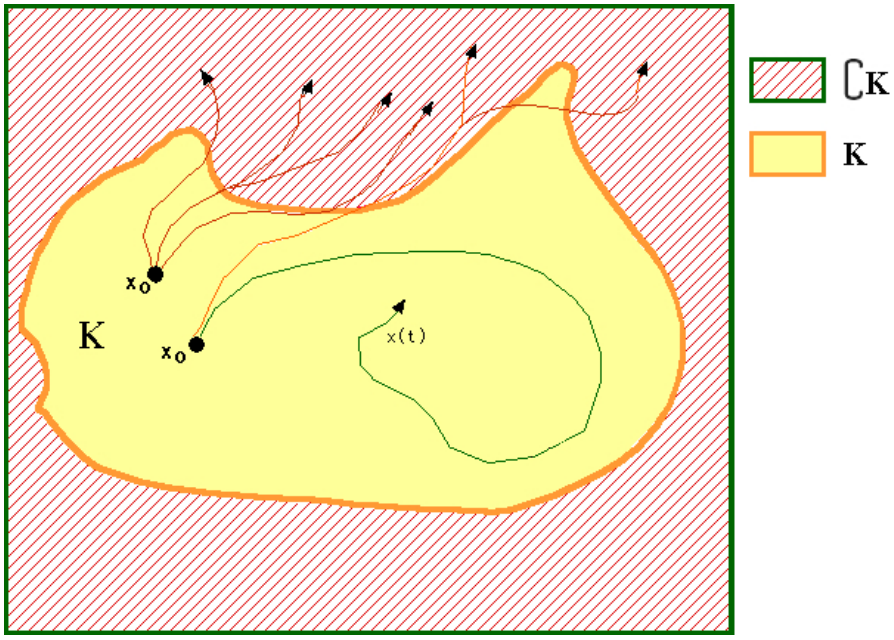
Viability Kernels
and
Capture Basins

Viability and Capturability Concepts

- **The viability kernel of the environment**: It is the subset (possibly empty) of the states in the environment from which starts at least a viable evolution (remaining all the time) in this environment,
- **The capture basin of the target viable in the environment**: It is the subset of the states in the environment from which starts at least one viable evolution in this environment until it reaches the target in finite time.

and it designs **retroactions** (feedbacks) which allow us to **pilot** the evolutions so as to maintain viability until, if any, capturing a target.

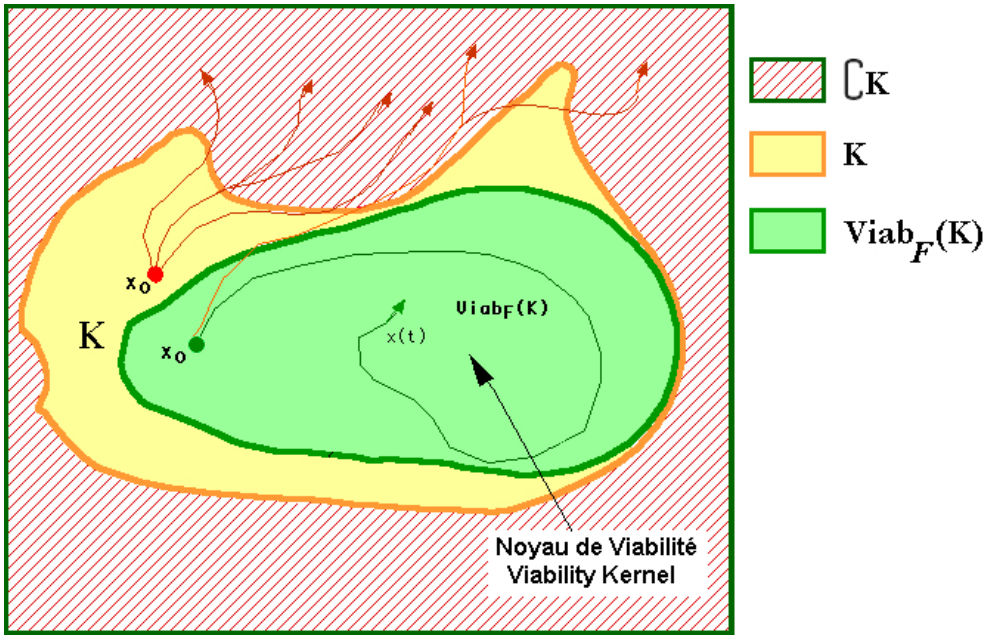
Viability and Invariance



K: Constraint Set.

1 Evolutions and Constraints.
An environment is described by a set of constraints K. The evolution is governed by a law f dependent of a parameter variable u.

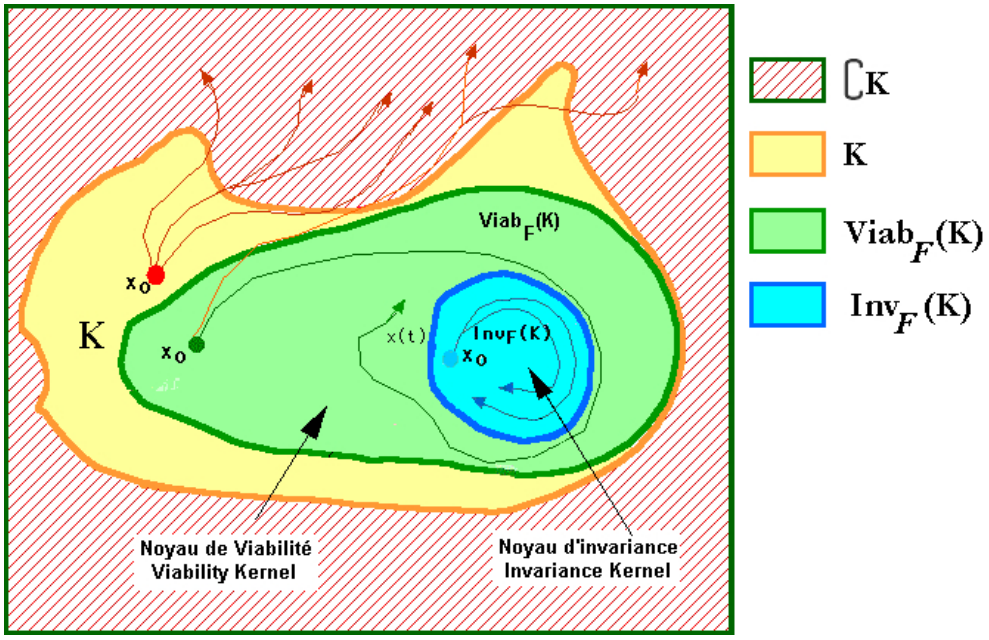
Viability and Invariance



2 Viability Kernel.
It is the set of the states in the environment K from which starts *at least* one evolution that remains always in K. All viable evolution remains necessarily always in $Viab_F(K)$.

$$Viab_F(K) := \{x \in K \mid \exists x(\cdot) \in \mathcal{S}_F(x),\ \forall t > 0,\ x(t) \in K\}$$

Viability and Invariance

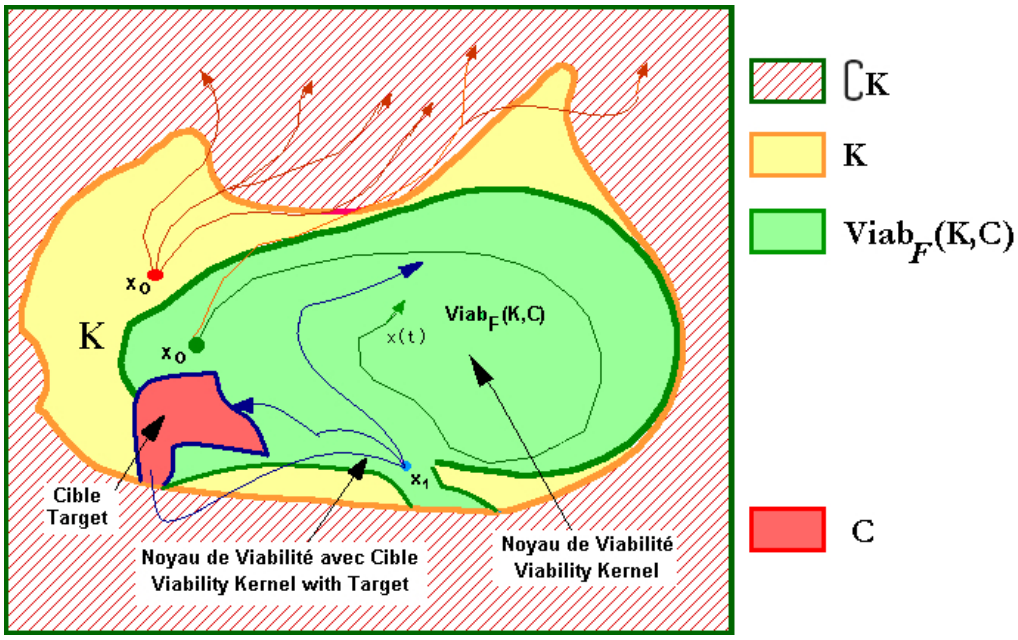


3 Viability Kernel and Invariance Kernel

The Invariance kernel is the set of the states in the environment K from which *all* evolution remain forever in the environment K. All invariant evolutions remain necessarily forever in $Inv_F(K)$.

$$Inv_F(K) := \{x \in K \mid \forall x(\cdot) \in \mathcal{S}_F(x),\ \forall t > 0,\ x(t) \in K\}$$

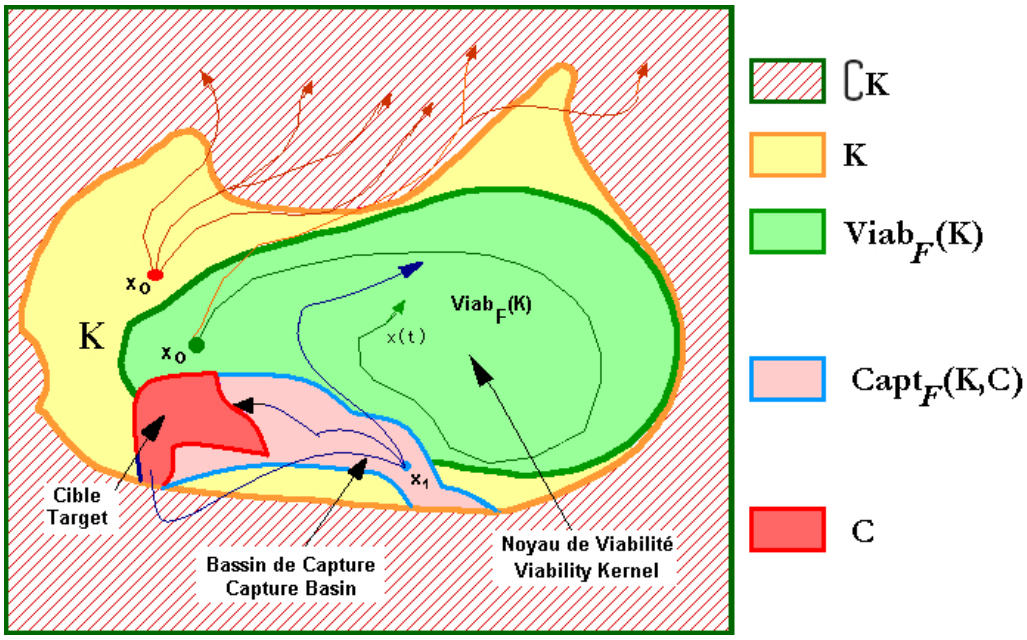
Capturability and Absorption



4 **Viability Kernel with Target.**
It is the set of the states in the environment K from which at least an evolution remains always in K whenever the target has not been reached. The viability kernel with target contains the viability kernel of the environment.

$$Viab_F(K,C) := \{x \in K \mid \exists x(\cdot) \in \mathcal{S}_F(x), \exists t^* > 0, \\ x(t^*) \in C,\ \forall t \in [0,t^*],\ x(t) \in K\} \cup Viab_F(K)$$

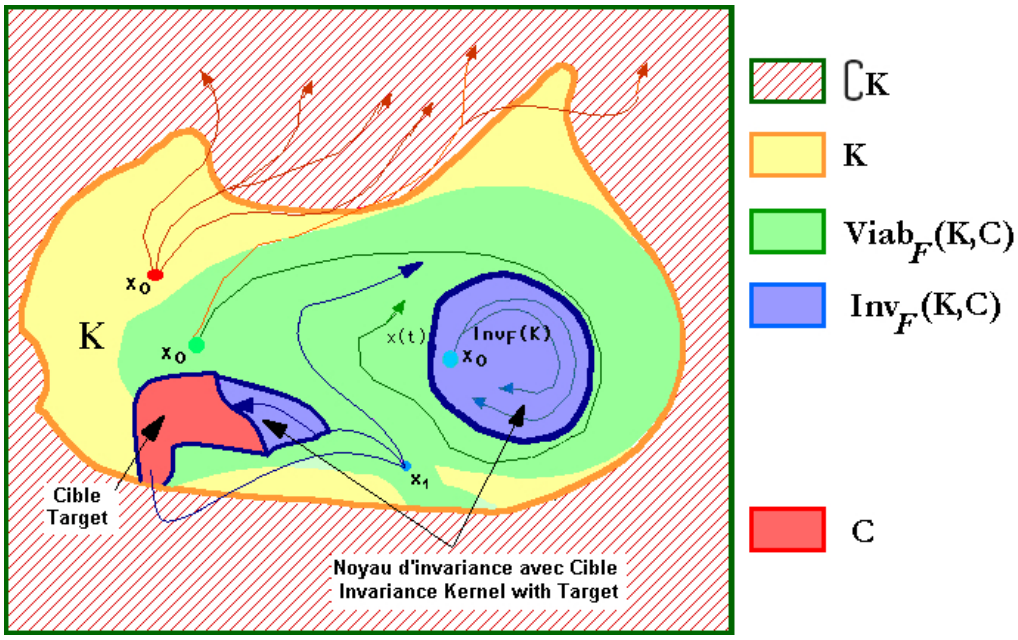
Capturability and Absorption



5 Capture Basin.
The basin of capture of the target, viable in an environment, is the subset (possibly empty) of the states of the environment K from which *at least* one evolution remains *viable* in the environment until it reaches the target *in finite time.* .

$$Capt_F(K,C) := \{x \in K \mid \exists x(\cdot) \in \mathcal{S}_F(x), \exists t^* > 0, \\ x(t^*) \in C,\ \forall t \in [0,t^*],\ x(t) \in K\}$$

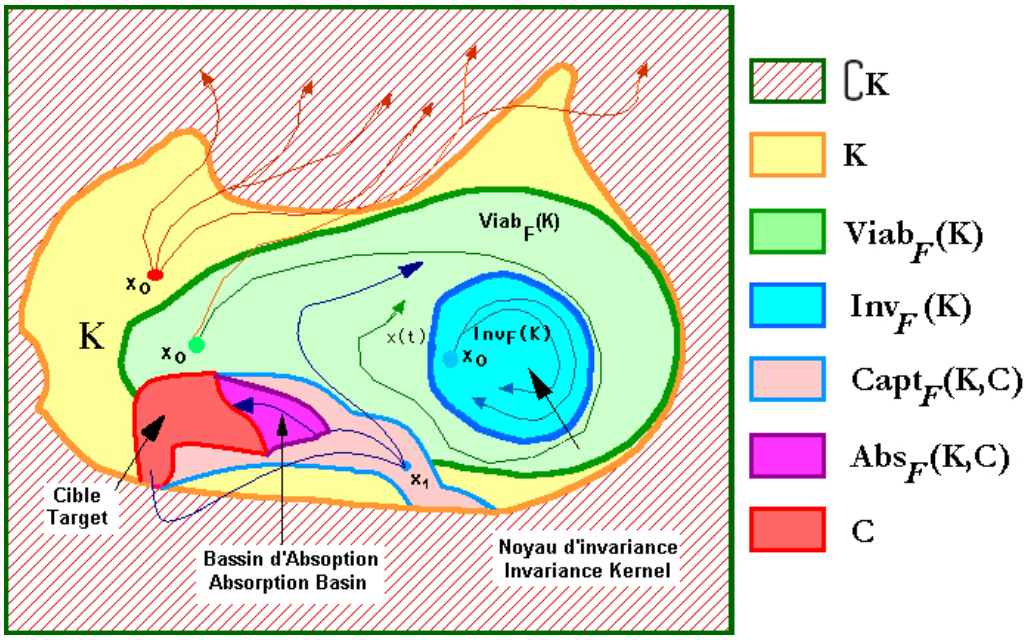
Capturability and Absorption



6 Invariance Kernel with Target.
The invariance kernel with target is the subset (possibly empty) of the states of the environment from which *all* evolutions leaving from this state remain in the environment whenever the target is not reached.

$$Inv_F(K,C) := \{x \in K \mid \forall x(\cdot) \in \mathcal{S}_F(x), \exists t^* > 0,\ x(t^*) \in C,\ \forall t \in [0,t^*],\ x(t) \in K \text{ or } \forall t > 0,\ x(t) \in K\}$$

Capturability and Absorption



7 Viable Absorption Basin.
The absorption basin of the target, viable in an environment, is the subset (possibly empty) of the states of the environment from which *all* evolutions leaving from this state reach the target *in finite time* without leaving the environment K.

$$Abs_F(K,C) := \{x \in K \mid \forall x(\cdot) \in \mathcal{S}_F(x), \exists t^* > 0,\ x(t^*) \in C,\ \forall t \in [0,t^*],\ x(t) \in K\}$$

Extension to Hybrid Constrained Dynamical Systems

Let $K \subset \mathbb{R}^n$ be a closed constraint set. Let $\mathcal{C} \subset K$ a closed target.

The aim is to reaching the target $\mathcal{C}$ before leaving K, or remaining in K if $\mathcal{C}$ cannot be reached.

An impulse (or hybrid) system is given by

$$F(x) := \{f(x,u)\} \text{ where } u \in U(x),$$

$$\Phi(x) := \begin{cases} \{\varphi(x,\pi) : \pi \in \Pi(x)\} & \text{if } x \in D(\Phi), \\ \emptyset & \text{if not,} \end{cases}$$

and for any evolution of the control $u(\cdot)$ the evolution of the state is a solution to the differential and/or the discrete inclusion:

$$x'(t) \in F(x(t), u(t)),$$
$$x^+ \in \Phi(x^-).$$

Extension to Hybrid Systems

Definition

We call run *of the impulse system* (F, Φ) *with initial condition* x_0 *associated to a control* $u(\cdot)$ *a finite or infinite sequence* $\{\tau_i, x_i, x_i(\cdot)\}_{i \in I}$ *in* $\mathbb{R}^+ \times \mathbb{R}^n \times S_F(\mathbb{R}^n)$, *where* $\{\tau_i\}_{i \in I}$ *is a sequence of times such that for all* $i \in I$

$$x_i'(t) = f(x_i(t), u(t)), \quad x_i(0) = x_i, \quad x_i(\tau_i) \in D(\Phi), \quad x_{i+1} \in \Phi(x_i(\tau_i))$$

We call trajectory *associated with the run, the function* $t \to x(t)$ *defined by*

$$x(t) = \begin{cases} x_0 & \text{if} \quad t < 0 \\ x_i(t - \sum_{j<i} \tau_i) & \text{if} \quad t \in \left(\sum_{j<i} \tau_j\right) + [0, \tau_i[\end{cases}$$

Extension to Hybrid Systems

Let $\mathcal{C}$ be a closed target to be reached in finite time satisfying $\mathcal{C} \cap D(\Phi) = \emptyset$. The Capture Basin of $\mathcal{C}$ is the domain of the minimal time-to-reach function and the epigraph of this function is the viability kernel of an extended dynamic (see Cardaliaguet, Quincampoix & S.-P., *Annals of the International Society of Dynamical Games, Birkhäuser, 1999*). This holds true in presence of impulses.

Assumptions (H)

(i) F is
- upper semi-continuous, non empty convex, compact valued and with linear growth,
- bounded on K,

i.e. $\exists M > 0,\ \forall x \in K,\ \forall y \in F(x),\ ||y|| \leq M.$

(ii) Φ is
- upper semi-continuous, non empty compact valued,
- such that $\forall x \in D(\Phi),\ \Phi(x) \cap D(\Phi) = \emptyset.$

Extension to Hybrid Systems

Theorem [S.-P., *HSCC 2001*, Crück & S.-P., *Set-Valued Analysis, 2003*]

Let F and Φ verifying (H). There exists a largest closed set, denoted $Hyb_{(F_C,\Phi)}(K)$, of all initial positions for which there exists at least a run viable in K whenever the target C is not reached.

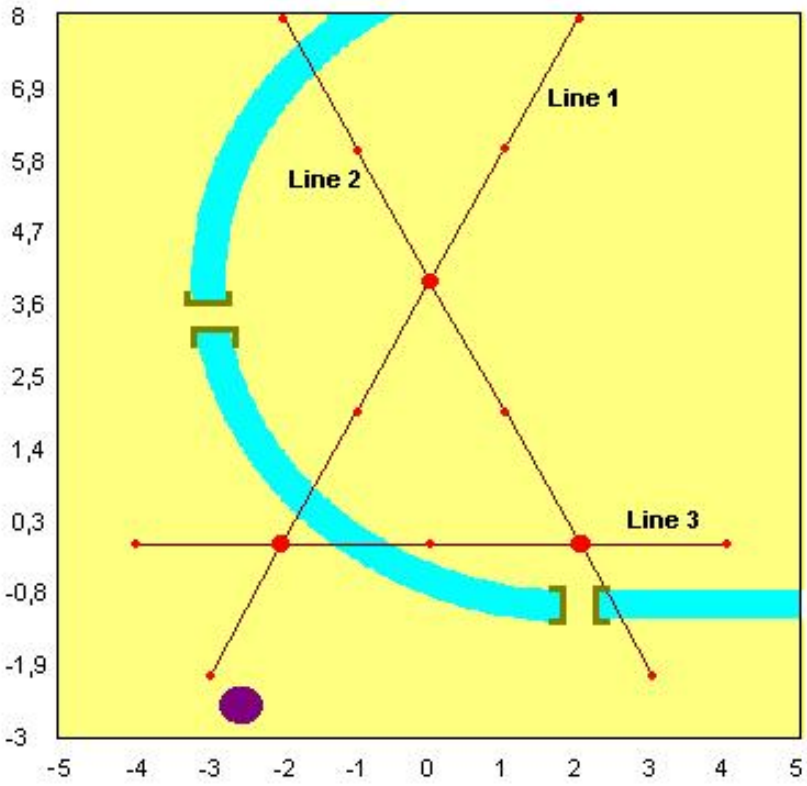
Consider the impulse extended system (Ψ, Ξ)

$$\begin{array}{lll} (x', z') & \in \Psi_T(x,z) := & \left\{ \begin{array}{l} F_C(x) \times \{-1\} \text{ si } x \notin C) \\ F_C(x) \times [0,1] \text{ si } x \in C) \end{array} \right. \\ (x^{n+1}, z^{n+1}) & \in \Xi(x^n, z^n) := & (\Phi(x^n), z^n) \end{array}$$

The minimal time-to-reach function of target C defined by $V(x_0) = \min\{z_0 \mid (x_0, z_0) \in Hyb_{(\Psi_C,\Xi)}(K \times R^+)\}$ is the lowest semi-continuous viscosity super solution of the H.J.B equation:

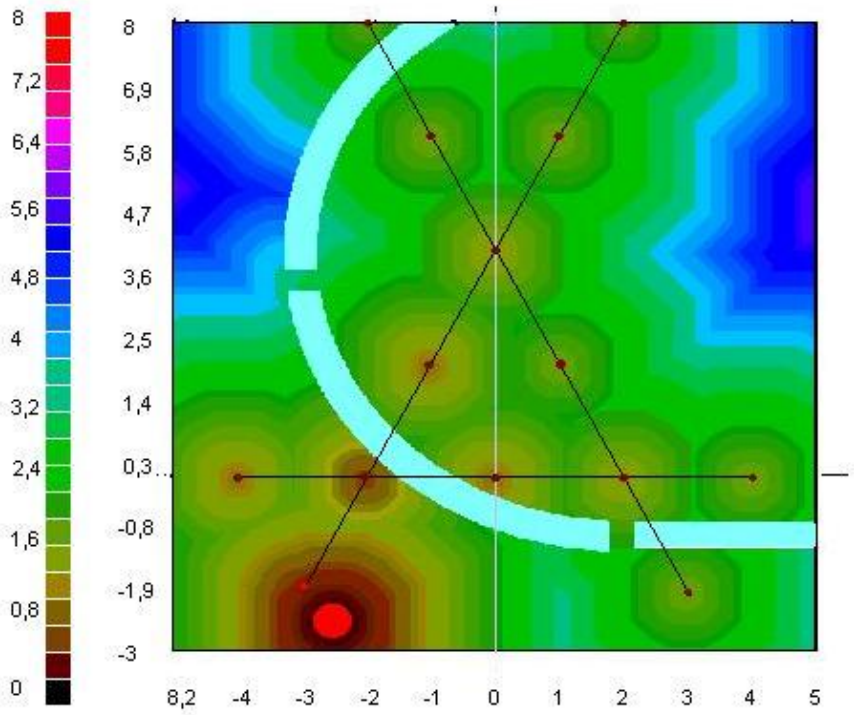
$$\forall\, x \in Dom(V) \backslash C,\ \max_{u \in U} < f(x,u), -\frac{d}{dx} V(x) > -1 = 0.$$

Example the “Metropolitan”



8 Minimal Time of an impulse system .

Example the “Metropolitan”



9 Minimal Time of an impulse system...

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   - *Mathematics: Fractal Sets and Attractors*
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Part II

Set Valued Numerical Analysis

and

Numerical Control Synthesis

Part III

Some Applications

to

Mathematics, Engineering,

and Hybrid Control

Mathematics : Approximation of Fractal Sets and Attractors

Approximation of Julia Sets and Fractals:

Pierre Fatou [1878-1929] and Gaston Julia [1893-1978] studied in depth the iterates of complex function

$$z \mapsto z^2 + u$$

or, equivalently, of the map

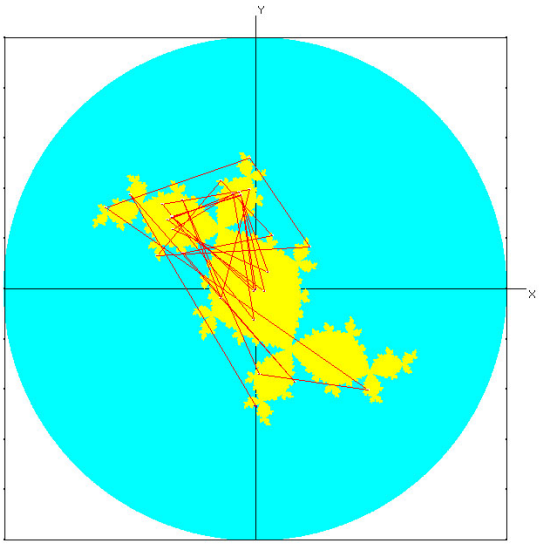
$$(x, y) \mapsto \varphi(x, y) := (x^2 - y^2 + a, 2xy + b)$$

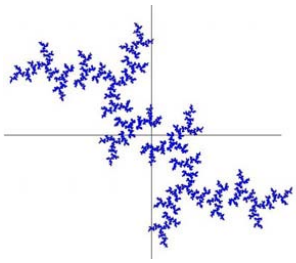
One can easily check that evolutions of iterates $z_{n+1} = z_n^2 + u$that are bounded must in fact remain in a ball or radius 2.

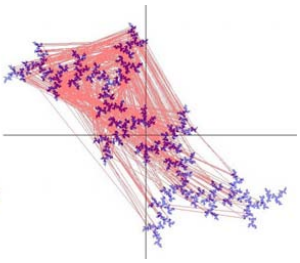
The subset $K_u := \mathrm{Viab}_\varphi(B(0, 2))$ is the *filled-in Julia set* and its boundary $J_u := \partial K_u$the *Julia set.*

Mathematics : Approximation of Fractal Sets and Attractors

Contrary to "shooting methods", the viability kernel algorithm provides the exact filled-in Julia sets and the viable iterates.
Examples of a filled-in Julia sets and of a Julia set, called Fatou dust.





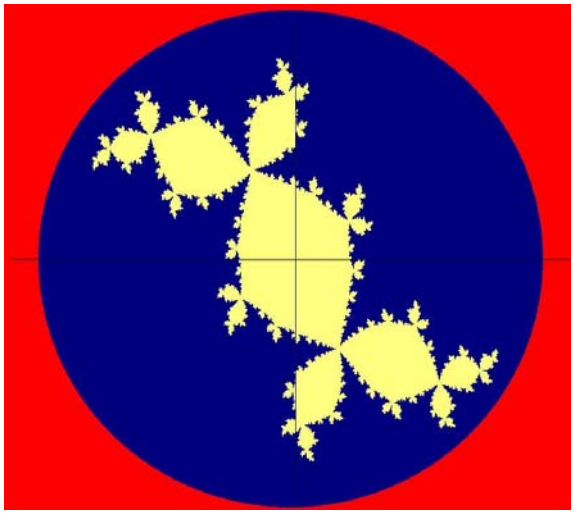


Fractal sets: Julia sets and Filled-in Julia sets

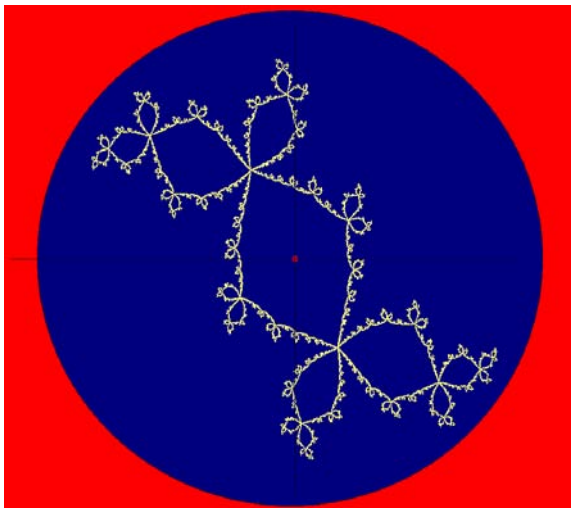
Theorem

The boundary $\partial \mathrm{Viab}_{\varphi}(K) = \mathrm{Viab}_{\varphi}(K \setminus C)$ ***of a viability kernel is the viability kernel of the complement of a subset*** $C \subset \mathrm{Viab}_{\varphi}(K)$ ***if and only if the boundary*** $\partial \mathrm{Viab}_{\mathcal{S}}(K)$ ***is viable and the interior*** $\mathrm{Int}(\mathrm{Viab}_{\mathcal{S}}(K))$ ***captures*** C.

So the viability kernel algorithm can be used to compute the Julia set:



Fill in Julia set



Julia set

Mathematics : Approximation of Fractal Sets and Attractors

Lorenz equations:

Lorenz introduced the following variables

① x, proportional to the intensity of convective motion,

② y, proportional to the temperature difference between ascending and descending currents,

③ z, proportional to the distortion (from linearity) of the vertical temperature profile.

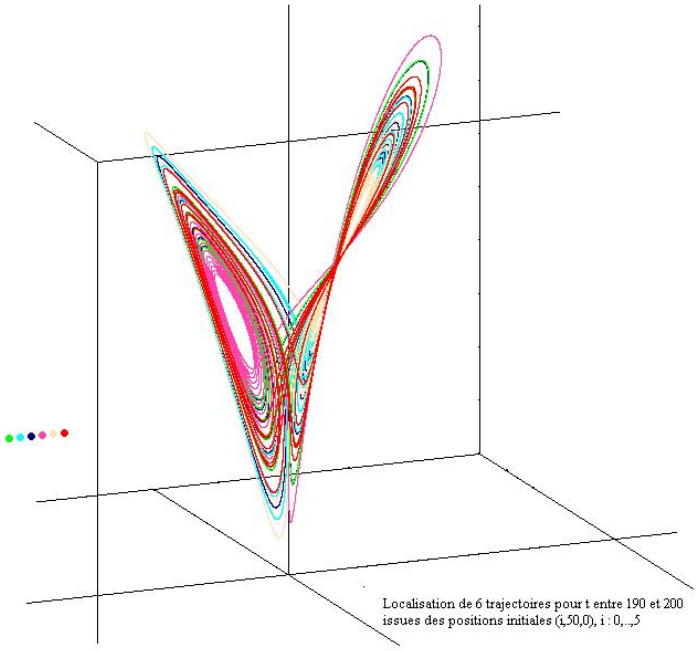
Mathematics : Approximation of Fractal Sets and Attractors

Their evolution is governed by the following system of differential equations:

$$\begin{cases} (i) & x'(t) = \sigma y(t) - \sigma x(t) \\ (ii) & y'(t) = rx(t) - y(t) - x(t)z(t) \\ (iii) & z'(t) = x(t)y(t) - bz(t) \end{cases}$$

where $\sigma > b + 1$. The vertical axis $(0, 0, z)_{z \in \mathbb{R}}$is a symmetry axis, which is also the viability kernel of the hyperplane $(0, y, z)$under the Lorenz system, from which the solutions are $(0, 0, ze^{-bt})$.

Some evolutions of the Lorenz system:



10 **Control Systems**

Trajectories of six evolutions starting from initial conditions $(i, 50, 0)$, $i = 0, \ldots, 5$.. *Only the part of the trajectories from step times ranging between 190 and 200 are shown for clarity.*

If the normalized Rayleigh number $r \in]0, 1[$, then 0 is an asymptotically stable equilibrium. If $r = 1$, the equilibrium 0 is "neutrally stable". When $r > 1$, the equilibrium 0 becomes unstable and two more equilibria appear:
$e_1 := \left(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1\right)$
$e_2 := \left(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1\right)$ They are stable when $1 < r^\star := \dfrac{\sigma(\sigma+b+3)}{\sigma-b-1}$ and unstable when $r > r^\star$.

Limit sets of evolutions and attractors:

Let $x(\cdot) \in \mathcal{C}(0,\infty;X)$ be an evolution. We say that subsets

$$\begin{cases} \omega(x(\cdot)) := \bigcap_{T>0} cl(x(\mathbb{R}_+)) = \text{Limsup}_{t\to+\infty}\{x(t)\} \\ \alpha(x(\cdot)) := \overleftarrow{\omega}(x(\cdot)) := \bigcap_{T>0} cl(x(\mathbb{R}_-) = \text{Limsup}_{t\to-\infty}\{x(t)\}) \end{cases}$$

of the cluster points when $t \to \infty$ and $t \to -\infty$ are respectively the *ω-limit set* of $x(\cdot)$ and the *α-limit set* of the evolution $x(\cdot)$.

Let $\mathcal{S}^K(x)$ the set of evolutions starting at $x \in K$ viable in K

$$\text{Attr}_{\mathcal{S}}(K) := \bigcup_{x(\cdot)\in\mathcal{S}^K(x), x\in K} \{\omega(x(\cdot))\} \text{ and } \text{Attr}_{\overleftarrow{\mathcal{S}}}(K) := \bigcup_{x(\cdot)\in\mathcal{S}^K(x), x\in K} \{\alpha(x(\cdot))\}$$

the ω-attractor or simply attractor and the α-attractor or *backward attractor* of the subset K under $\mathcal{S}$ respectively.

Viability Properties of Limit Sets and Attractors:

If $\mathcal{S}$ is upper semicompact, the ω-limit set $\omega(x(\cdot))$ of an evolution $x(\cdot) \in \mathcal{S}(x)$ is always forward and backward viable under $\mathcal{S}$:

$$\omega(x(\cdot)) \;=\; \mathrm{Viab}_{\mathcal{S}}(\omega(x(\cdot))) \;=\; \mathrm{Viab}_{\overleftarrow{\mathcal{S}}}(\omega(x(\cdot)))$$

The forward and backward attractors of K under $\mathcal{S}$, as well as their closures are respectively subsets viable and backward viable under the evolutionary system:

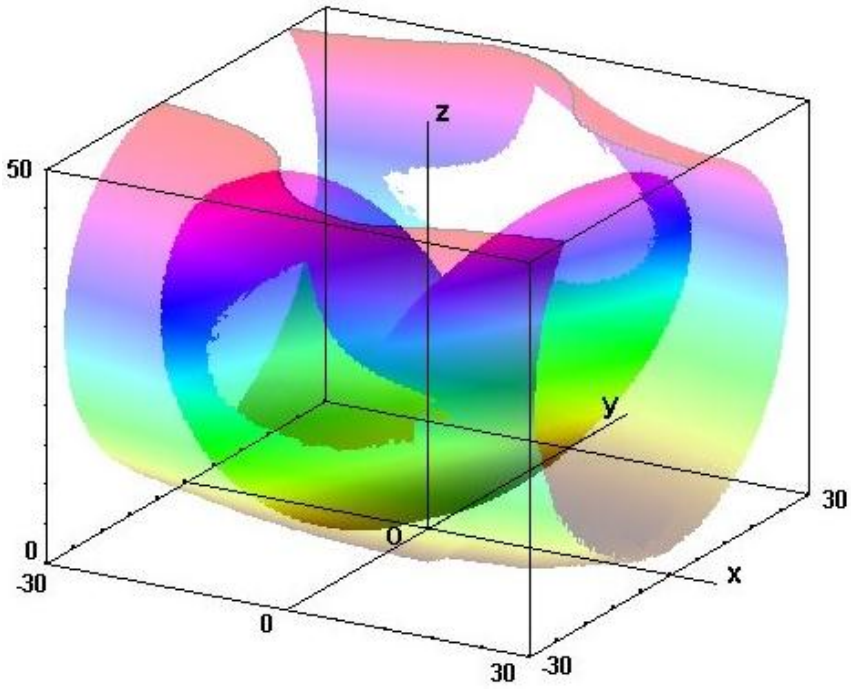
$$\mathrm{Attr}_{\mathcal{S}}(K) \;=\; \mathrm{Viab}_{\mathcal{S}}(\mathrm{Attr}_{\mathcal{S}}(K)) \;=\; \mathrm{Viab}_{\overleftarrow{\mathcal{S}}}(\mathrm{Attr}_{\mathcal{S}}(K))$$

and

$$\mathrm{Attr}_{\overleftarrow{\mathcal{S}}}(K) \;=\; \mathrm{Viab}_{\mathcal{S}}(\mathrm{Attr}_{\overleftarrow{\mathcal{S}}}(K)) \;=\; \mathrm{Viab}_{\overleftarrow{\mathcal{S}}}(\mathrm{Attr}_{\overleftarrow{\mathcal{S}}}(K))$$

Attractor and Fractal sets

Forward and Backward Viability Kernel:



11 **The viability kernel of a cube K in $\mathbb{R}^3$ of the forward and backward Lorenz system.** *We take $\sigma = 10$, $b = \frac{8}{3}$ and $r = 28$. Whenever the backward viability kernel is contained in the interior of K, the backward viability kernel is contained in the forward viability kernel. The color scale provides the third coordinates. (computed with the VKA.)*

Attractor and Fractal sets

Localization of attractors:

They are consequently contained in the intersection of the viability kernel of K and the backward viability kernel of K:

$$\mathrm{Attr}_{\mathcal{S}}(K) \cup \mathrm{Attr}_{\overleftarrow{\mathcal{S}}}(K) \subset \mathrm{Viab}_{\mathcal{S}}(K) \cap \mathrm{Viab}_{\overleftarrow{\mathcal{S}}}(K)$$

Furthermore:

$$\mathrm{Attr}_{\mathcal{S}}(K \backslash \mathrm{Viab}_{\overleftarrow{\mathcal{S}}}(K)) \subset \mathrm{Viab}_{\mathcal{S}}(K) \cap \partial \mathrm{Viab}_{\overleftarrow{\mathcal{S}}}(K)$$

Theorem

If $\mathrm{Viab}_{\overleftarrow{\mathcal{S}}}(K) \subset \mathrm{Int}(K)$, then $\mathrm{Attr}_{\mathcal{S}}(K) \subset \mathrm{Viab}_{\overleftarrow{\mathcal{S}}}(K) \subset \mathrm{Inv}_{\mathcal{S}}(K)$

Let us consider two subsets K and $L \subset K$. Then

$$\mathrm{Attr}_{\mathcal{S}}(K) \subset \mathrm{Attr}_{\mathcal{S}}(L) \cup \overline{\complement \mathrm{Capt}_{\mathcal{S}}(K, \mathrm{Viab}_{\mathcal{S}}(L))}$$

Fluctuation Basin:

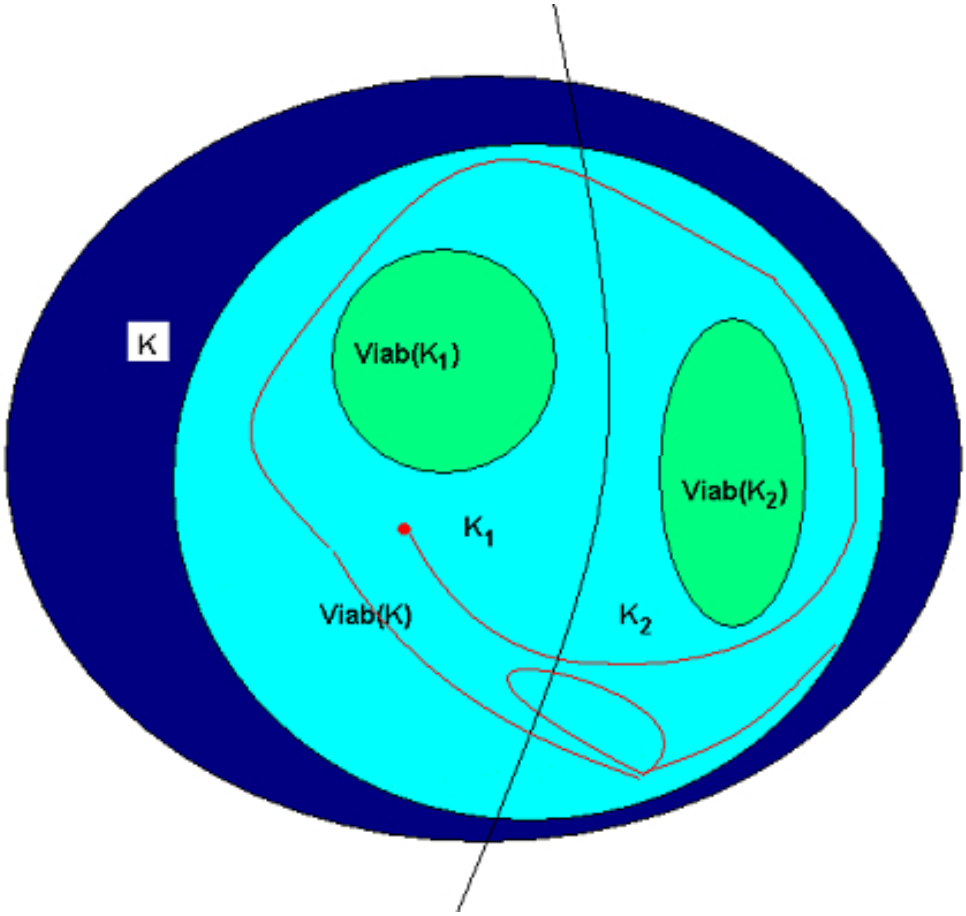
Let $K_1 \subset K$ and $K_2 \subset K$ be two closed subsets covering K: $K = K_1 \cup K_2$. The *fluctuation basin* $\mathrm{Fluct}(K_1, K_2)$ between K_1 and K_2 is the subset of initial states $x \in K$ from which all evolutions $x(\cdot) \in \mathcal{S}(x)$ viable in K fluctuate back and forth between K_1 to K_2 in the sense that the evolution leaves successively K_1 and K_2 in finite time.

Theorem 0.0.1 *The fluctuation basin is equal to the complement*

$$\mathrm{Fluct}(K_1, K_2) := \complement(\mathrm{Capt}_{\mathcal{S}}(K_1 \cup K_2, \mathrm{Viab}_{\mathcal{S}}(K_1) \cup \mathrm{Viab}_{\mathcal{S}}(K_2)))$$

of the capture basin of the union $\mathrm{Viab}_{\mathcal{S}}(K_1) \cup \mathrm{Viab}_{\mathcal{S}}(K_2)$.
If we assume furthermore that $K_i \subset \mathrm{Int}(K_i)$, $i = 1, 2$, *then*

$$\mathrm{Fluct}(K_1, K_2) := \complement(\mathrm{Viab}_{\mathcal{S}}(K_1) \cup \mathrm{Viab}_{\mathcal{S}}(K_2))$$


K
Viab(K1)
Viab(K2)
K1
Viab(K)
K2

Engineering : Designing discontinuous control for the Double Integrator with Target

(with the collaboration of Meeko Oishi, Univ. British Columbia)

Let us consider the following dynamics:

$y"(t) = u \in \mathcal{U} := [u_{min}, u_{max}] = [-4, 4]$

One wants to stabilize the state in minimal and finite time at a neighborhood of an equilibrium. Let us consider the constraints

$x(t) = (y(t), y'(t)) \in K := [y_{min}, y_{max}] \times [y'_{min}, y'_{max}]$

The system becomes

$$\begin{aligned} x_1'(t) &= x_2(t) \\ x_2'(t) &= u \\ \vartheta'(t) &= -1 \end{aligned}$$

The constraint set is $K := [-5, 5] \times [-5, 5]$, and the target is $\mathcal{C} := \mathrm{B}((0, 0), 0.2)$.

Engineering : Designing discontinuous control

We successively compute

1 - the viability kernel of $K \times \mathbb{R}^+$ which is the epigraph of the minimal time function

2 - the viability kernels for 3 linearized feedbacks

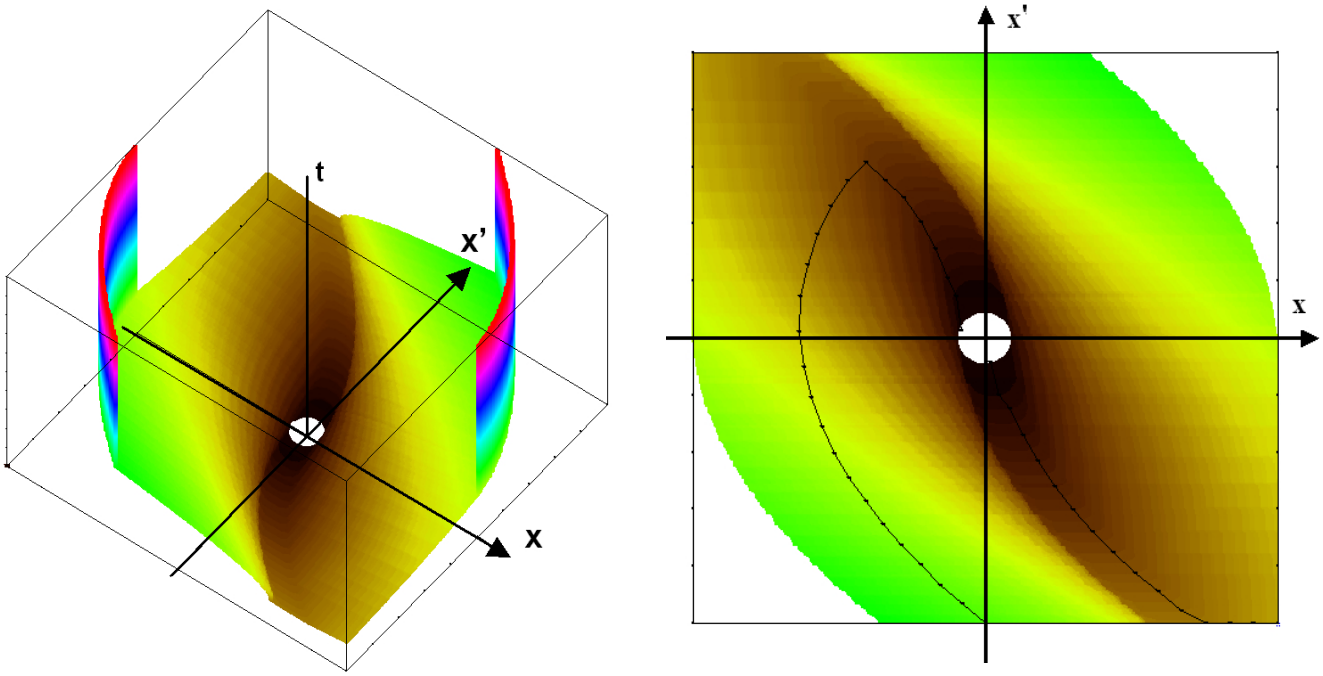
$$u(x, \beta^i) = -\beta_1^i y - \beta_2^i y'$$

(of "P.I.D." type),

3 - the impulse viability kernel with switching controls

$$\{u(x, \beta^1), u(x, \beta^2), u(x, \beta^3)\}.$$

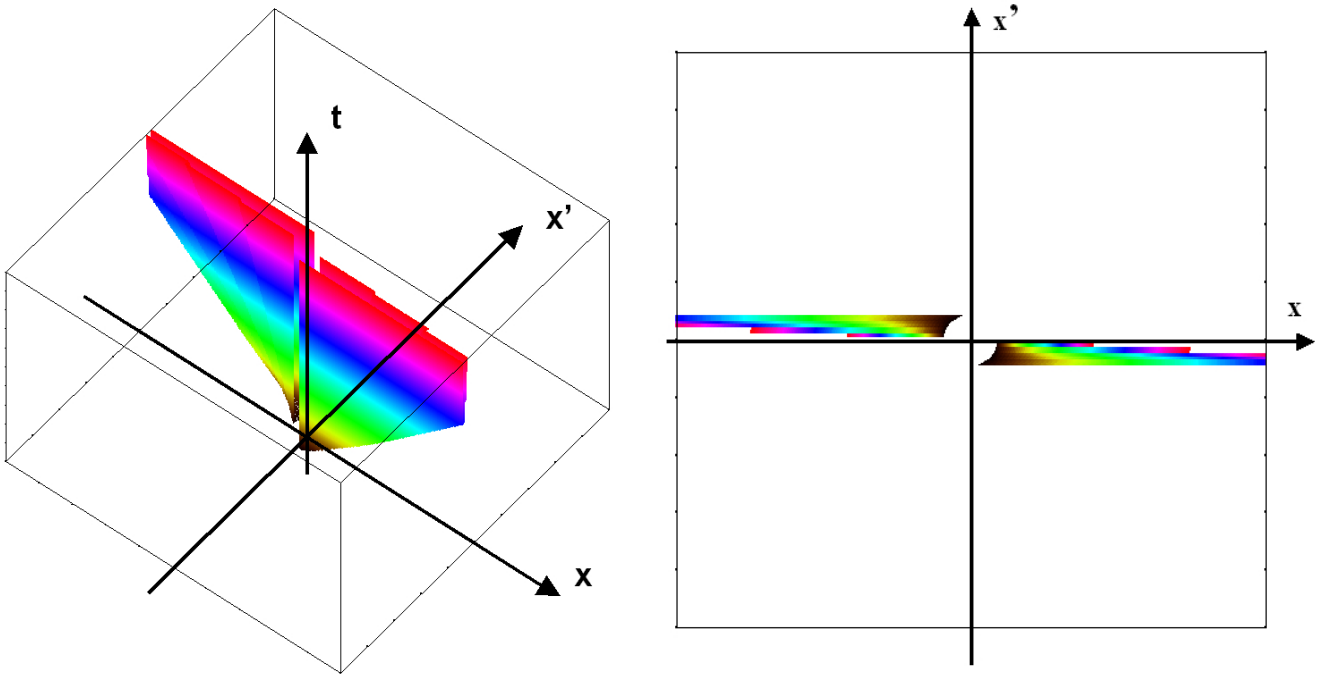
The minimal time function : The Continuous Case



Graph of the minimal time function for the full controlled system

$$y'' = u \in [-4, 4]$$

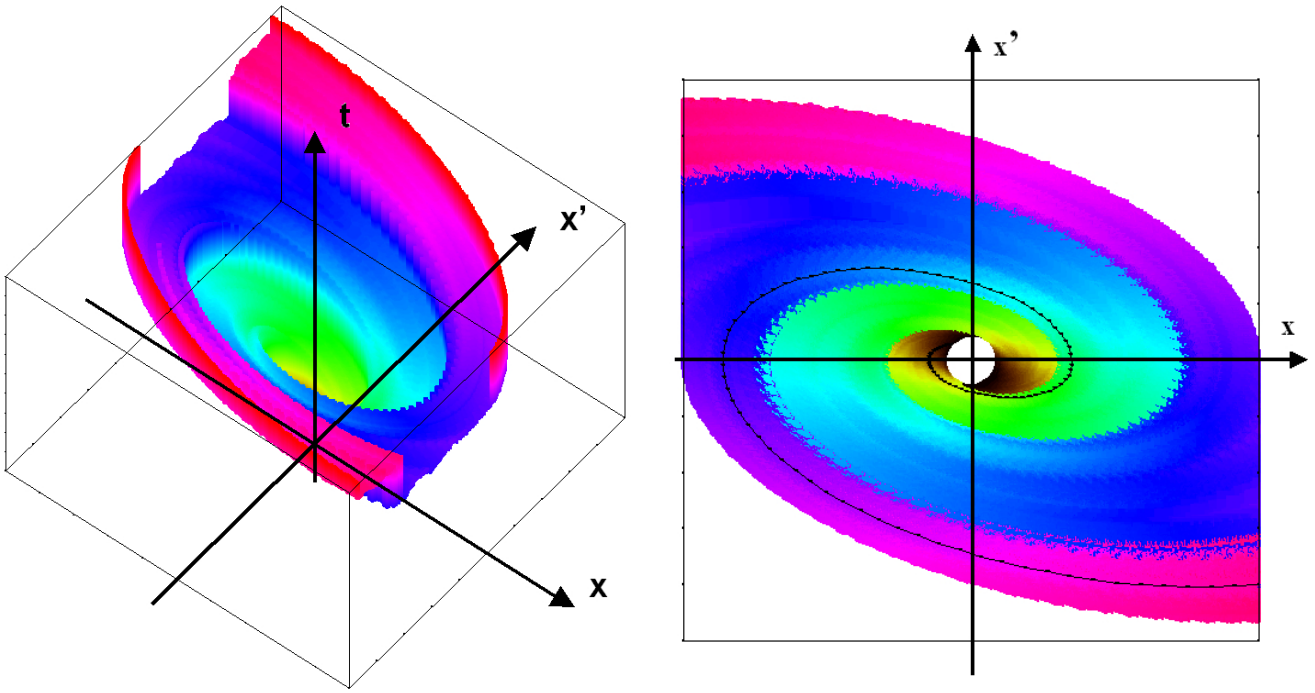
The minimal time function in mode 0



Graph of the minimal time function

$$y'' = u(x, \beta^1) = 0$$

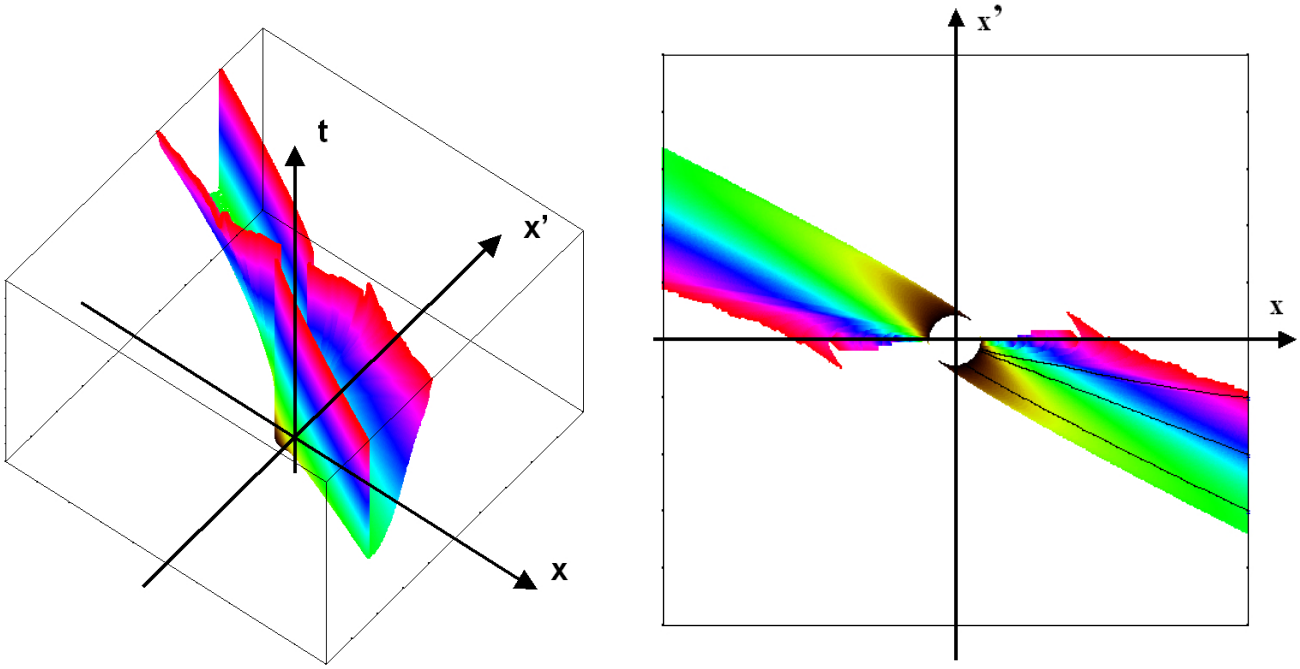
The minimal time function in mode 1



Graph of the minimal time function

$$y'' = u(x, \beta^3) = -0.3y - 0.3y'$$

The minimal time function in mode 2

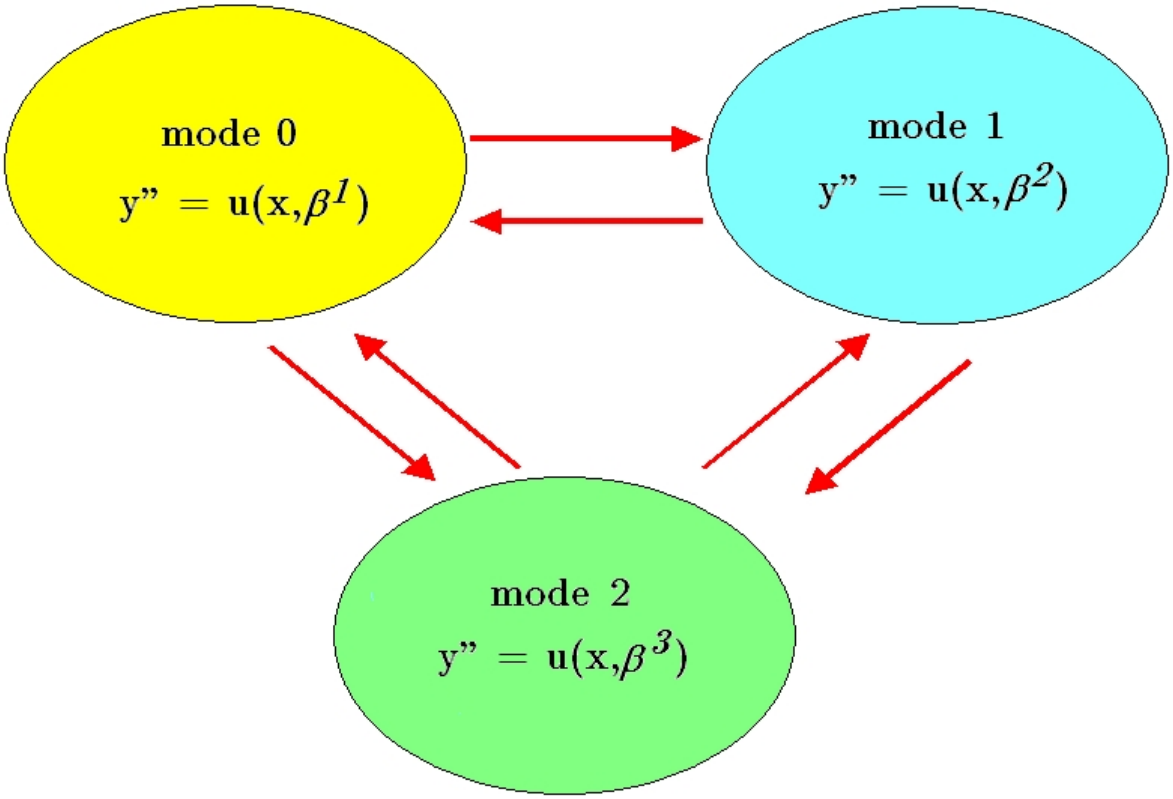


Graph of the minimal time function

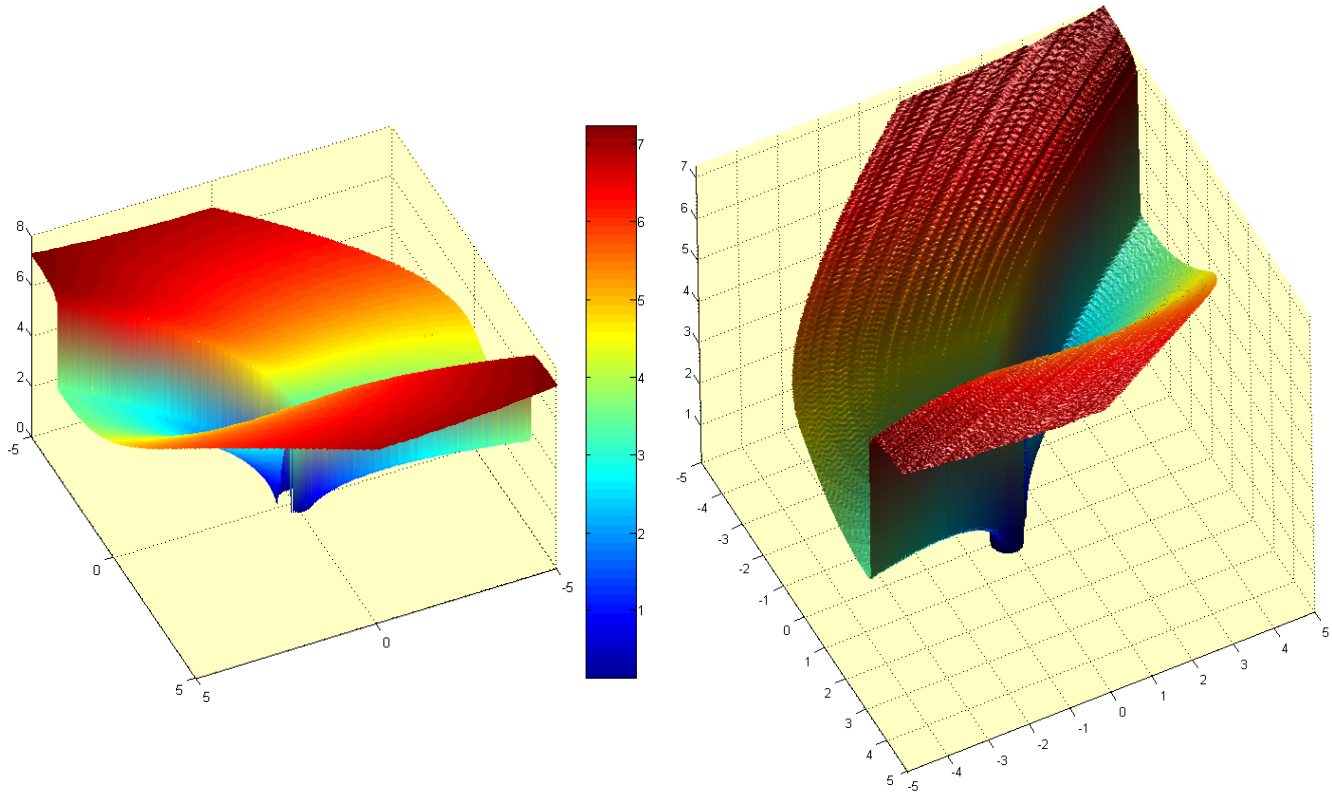
$$y'' = u(x, \beta^2) = -0.122y - 0.7y'$$

The minimal time function : The Hybrid Case

Switching modes are now authorized:



The minimal time function : The Hybrid Case

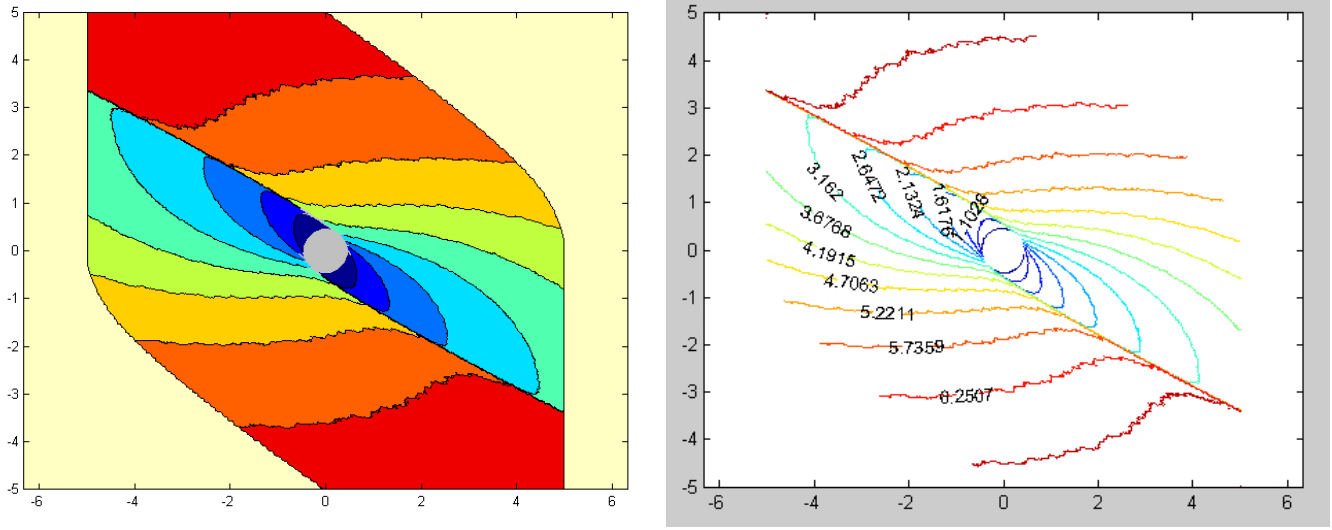


Graph of the minimal time function

$$y" \in \{u(x, \beta^i),\ i = 1, 2, 3\}$$

Minimal Time with Switching Controls

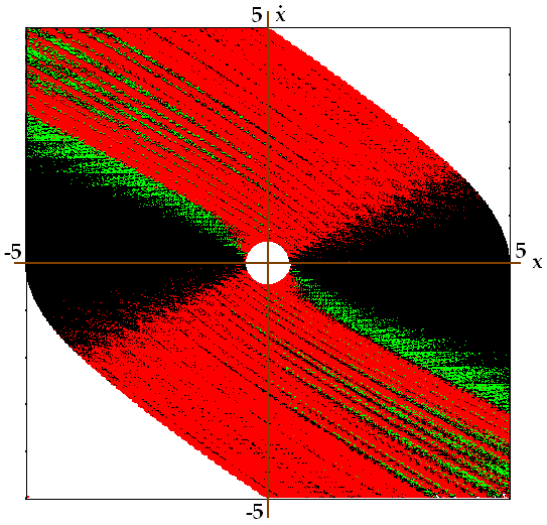
Iso Level Curves for the Minimal Time Function

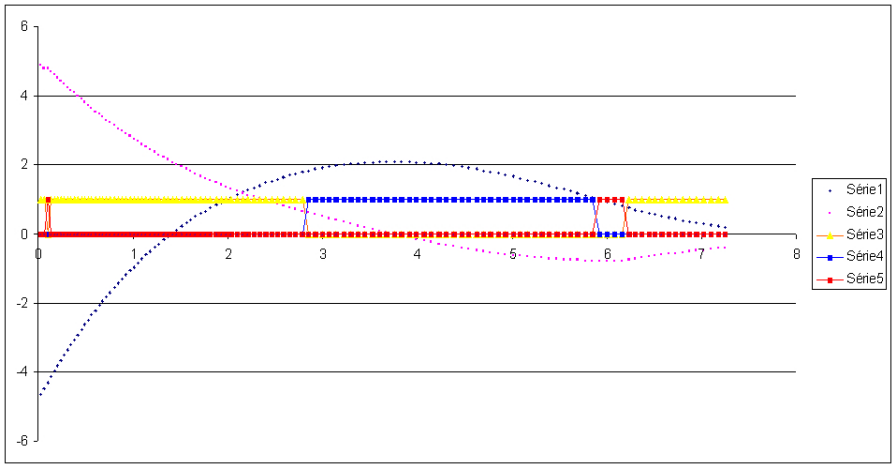


Iso level curves of the minimal time function for the full hybrid control system $y" \in \{u(x, \beta^i),\ i = 1, 2, 3\}$

Approximation of the Switching Strategy

in red, optimal feedback control is $u(x, \beta^2)$,
in black, optimal feedback control is $u(x, \beta^1)$
in green, optimal feedback control is $u(x, \beta^0)$.





Temporal history of the impulse control, of y and of y'.

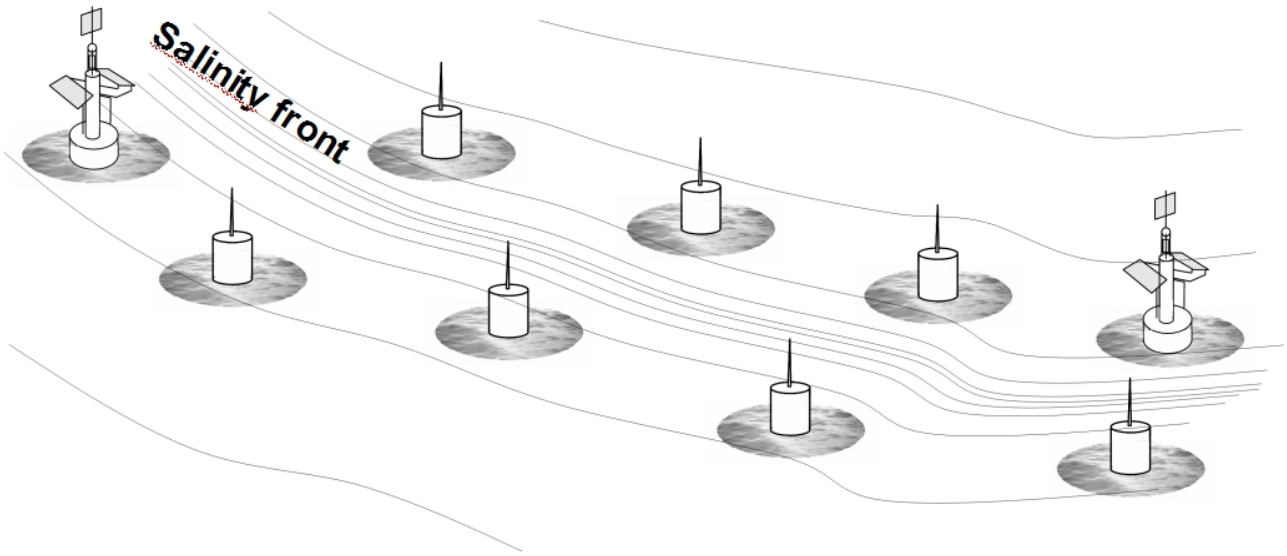
Embedded Systems: Lagrangian Sensors

(in collaboration: Alexandre Bayen, Andrew Tinka, Qingfang Wu)

Operational mission: feature tracking

Fleet of active autonomous drifters tracking:

- Salinity fronts
- Turbidity plumes



Lagrangian Sensors Network

Application for the Californian Dept of water Ressources.

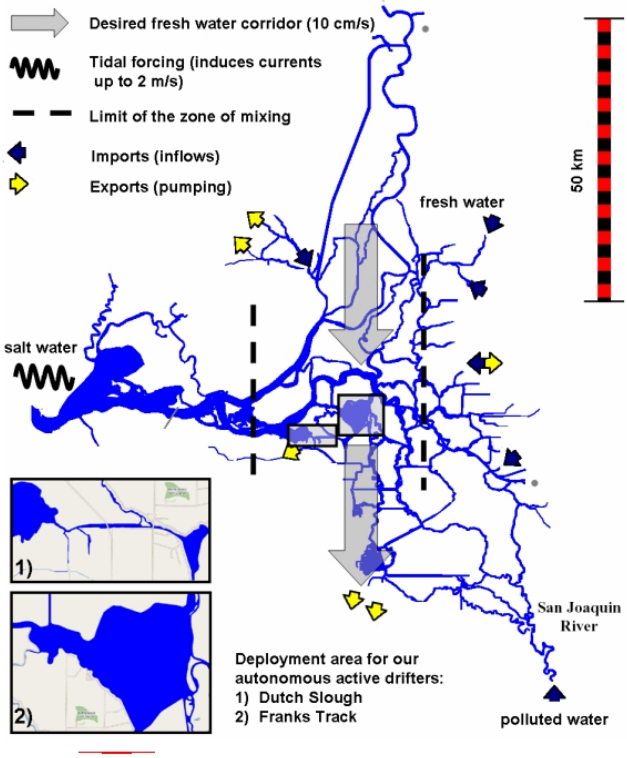
Vision: fresh water corridor

« Manage » the mixing

- Monitor salinity
- Track salinity fronts
- Transmit data in real time

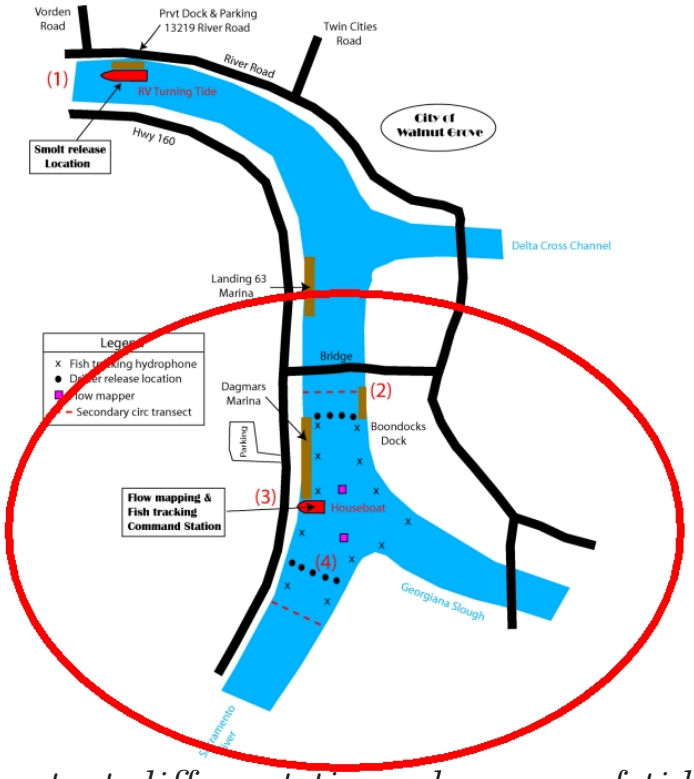
⇒ **Goal 1:**
Help the state of California build infrastructure

⇒ **Goal 2:**
Monitor salinity in real-time (to help operate the infrastructure)



Lagrangian Sensors Network

Georgianna Slough data collection



Testing site

- *pick the drifter up from the water,*
- *drive it up and dump it again,*
- *identifying the current at different times because of tidal forcing (hybrid system) and developping viability based software.*

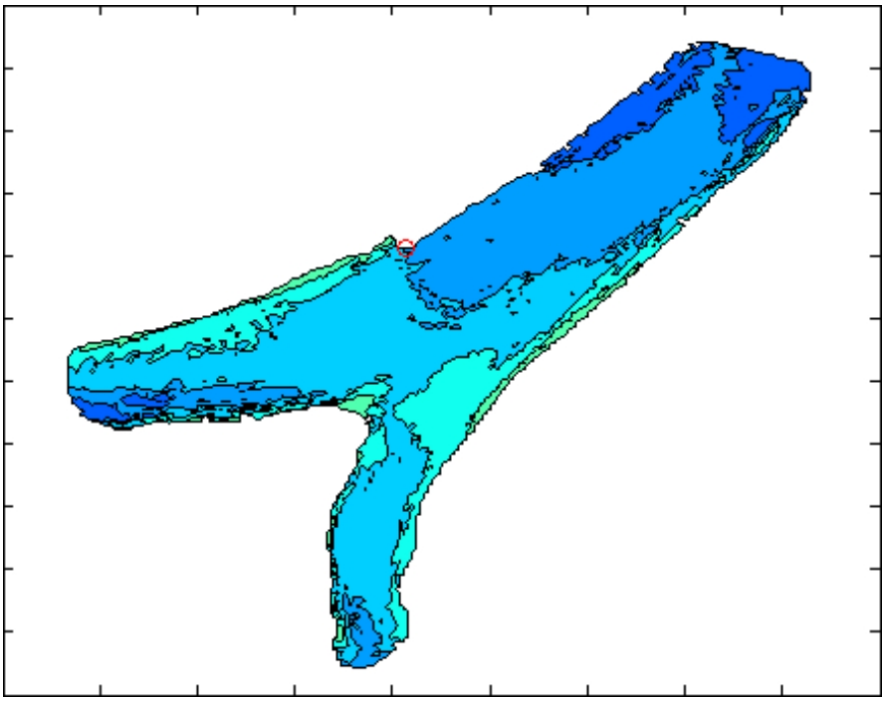
Lagrangian Sensors Network

12 Launching drifters .

In collaboration with

Prof. Mark Stacey
DWR
and
USGS.

Lagrangian Sensors Network



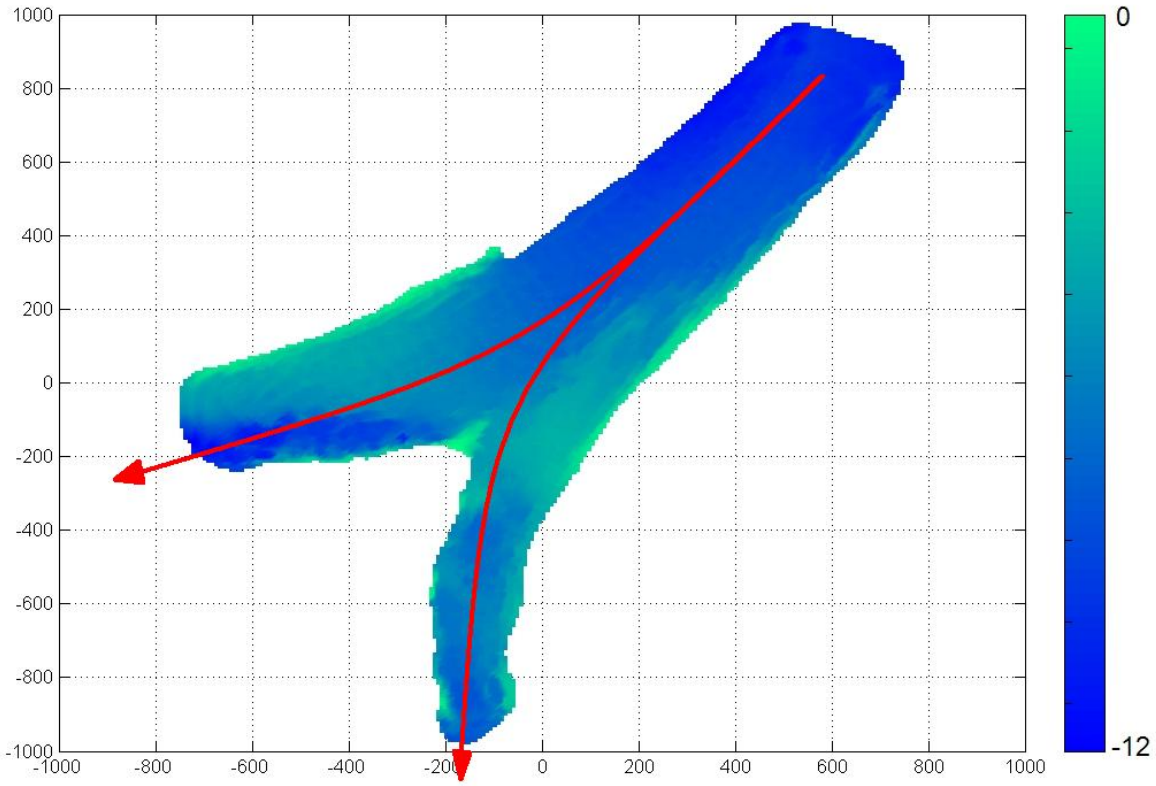
13 measuring the flow using GPS localisation:

Trajectories of drifters picked-up by boat before they exit the constraint domain.

Operational goals:

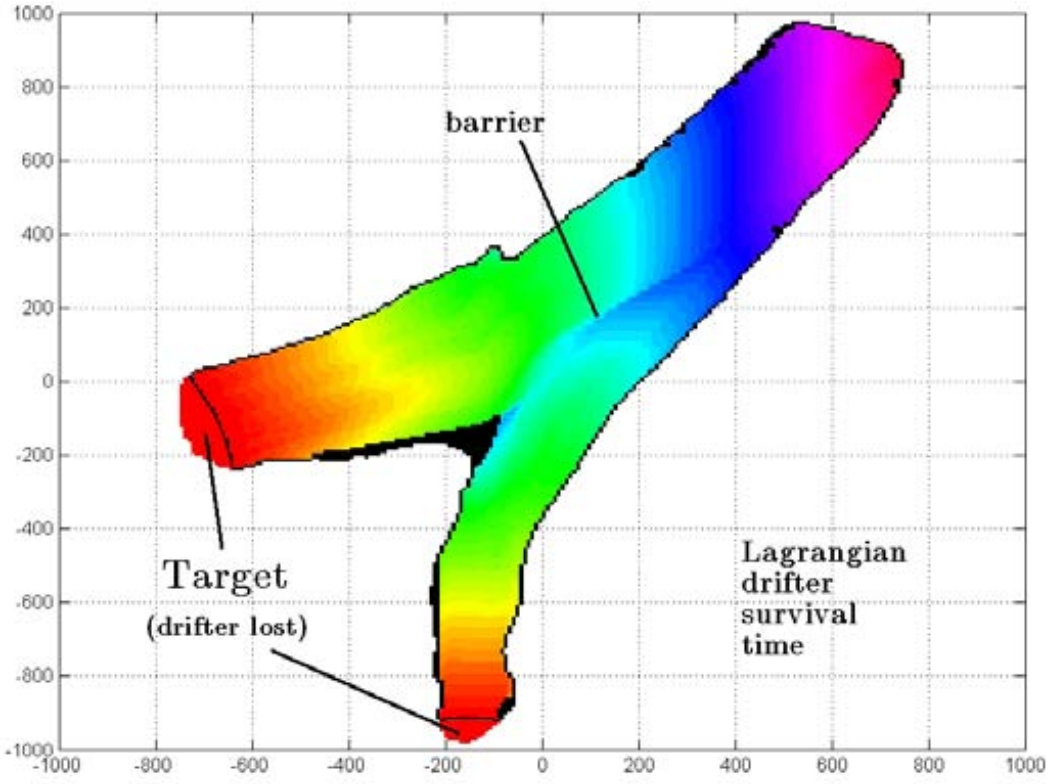
- data assimilation, inverse modeling,
- hydrodynamic parameter estimation (current, discharge, depth, friction, etc...)

Lagrangian Sensors: Computing Minimal Time

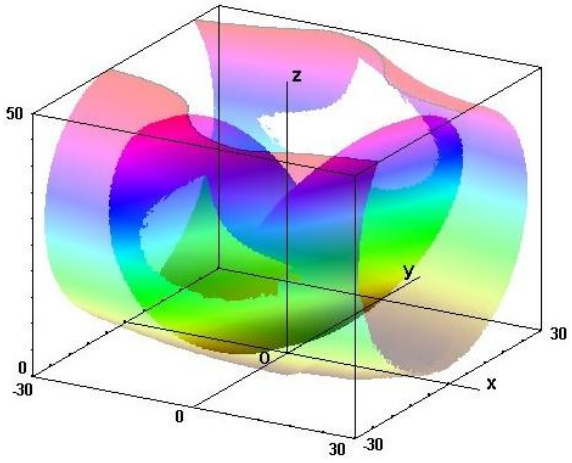
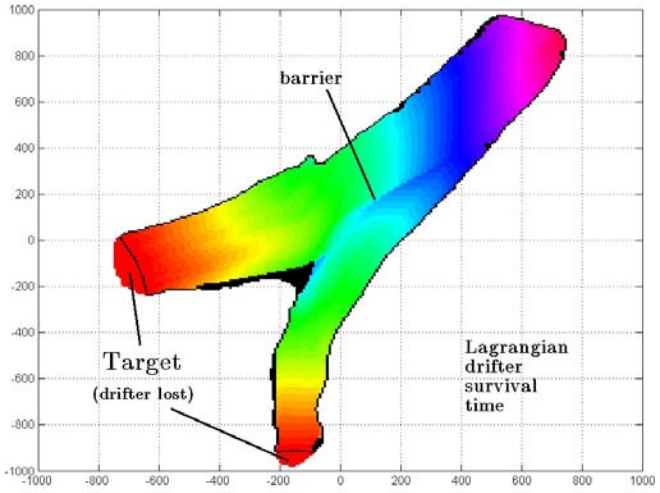
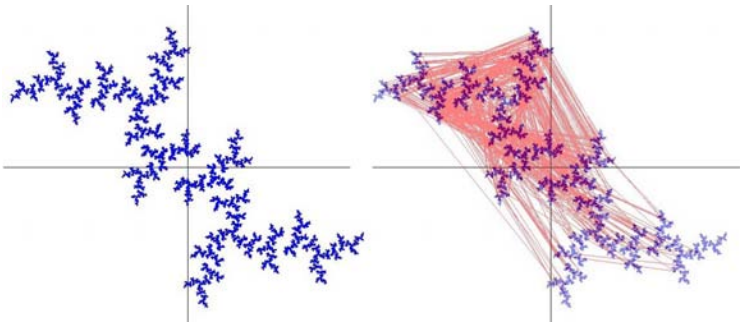


The constraint set.

Lagrangian Sensors: Computing Minimal Time



Open problems:

- fast computation of hybrid kernels and hybrid minimum time function,
- handling data uncertainty using robust viability,
- obtaining over/under approximations of viability, and reachable sets
- analyzing viability of networks of dynamical systems.

Thank you for your attention