On the synthesis of correct-by-design embedded control software

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Introduction

Examples of networked embedded control systems
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How are embedded control systems designed today?
This iterative scheme has several drawbacks:

- Validation by extensive simulation and testing increases our confidence in the software but fails to provide adequate guarantees of correct operation and performance;
- Formal verification is currently limited to finite state systems and thus cannot be used to verify properties depending on continuous components;
- Extensive validation is time consuming thus increasing the cost and time-to-market of embedded software.
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- Extensive validation is time consuming thus increasing the cost and time-to-market of embedded software.

Some of these disadvantages can be mitigated by adopting a *correct-by-design* approach to the development of embedded control software.
I shall adopt a three step approach to the synthesis of correct-by-design embedded control software.
Correct-by-design synthesis
A three step approach

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\[ x(t+1) = f(x(t), u(t)) \]
\[ \frac{dx(t)}{dt} = f(x(t), u(t)) \]

Continuous dynamics
Finite bisimulation
Hardware+software
Discrete controller
Correct-by-design synthesis
A three step approach

I shall adopt a three step approach to the synthesis of correct-by-design embedded control software.

\[
x(t+1) = f(x(t), u(t)) \\
dx(t)/dt = f(x(t), u(t))
\]

continuous dynamics

finite bisimulation

hardware+software

\[
q(t+1) = g(q(t), x(t)) \\
u = k(q(t), x(t))
\]

hybrid controller

discrete controller
Correct-by-design synthesis
A three step approach

Ultimately, I would like to:

1. Specify the continuous dynamics;
2. Specify the software+hardware platform;
3. Define the specification;
4. Obtain embedded code enforcing the specification for the continuous dynamics on the given software+hardware platform.
Correct-by-design synthesis
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This is a long term goal. Nevertheless, several key ingredients of the proposed approach are already available. In this talk I will focus on one such ingredient:

Existence of finite approximate bisimulations for control systems.
Key ingredients
Control systems as transition systems

Definition
A transition system is a quintuple \( T = (Q, L, \rightarrow, O, H) \), consisting of:
- A set of states \( Q \);
- A set of inputs \( L \);
- A transition relation \( \rightarrow \subseteq Q \times L \times Q \);
- An output set \( O \);
- An output function \( H : Q \rightarrow O \).
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Control systems as transition systems

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- An output function $H : Q \to O$.

\[
\begin{align*}
a &: = 4 \\
b &: = 1 \\
\text{while } a > 0 \\
a &: = a + b \\
\text{end while}
\end{align*}
\]
Can we regard control systems as transition systems?

**Definition**

A *control system* is a quadruple $\Sigma = (\mathbb{R}^n, U, \mathcal{U}, f)$, where:

- $\mathbb{R}^n$ is the state space;
- $U \subseteq \mathbb{R}^m$ is the input space;
- $\mathcal{U}$ is “nice” subset of the set of all functions of time from intervals of the form $]a, b[$ to $U$ with $a < 0$ and $b > 0$;
- $f : \mathbb{R}^n \times U \rightarrow \mathbb{R}^n$ is a “nice” continuous map.
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- $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ is a “nice” continuous map.

A “nice” curve $\mathbf{x} : ]a, b[ \to \mathbb{R}^n$ is said to be a *trajectory* of $\Sigma$ if there exists $\mathbf{u} \in \mathcal{U}$ satisfying $\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t))$, for almost all $t \in ]a, b[$.
Given a control system $\Sigma = (\mathbb{R}^n, U, \mathcal{U}, f)$ and sampling time $\tau \in \mathbb{R}^+$, define the transition system:

$$T_{\tau}(\Sigma) := (Q, L, \rightarrow, O, H),$$

where:

- $Q = \mathbb{R}^n$;
- $L$ is the set of all the curves in $\mathcal{U}$ of duration $\tau$;
- $q \xrightarrow{u} p$ if $x(\tau, q, u) = p$;
- $O = \mathbb{R}^n$;
- $H = 1_{\mathbb{R}^n}$.

The output set $O = \mathbb{R}^n$ is equipped with the metric $d(p, q) = \|p - q\|$. 
Given a control system $\Sigma = (\mathbb{R}^n, U, \mathcal{U}, f)$ and sampling time $\tau \in \mathbb{R}^+$, define the transition system:

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Can we replace $T_\tau(\Sigma)$ with an equivalent and yet finite transition system?
The usual notion of (bi)simulation requires exact matching of outputs.

**Definition**

Let $T_1 = (Q_1, L_1, O, H_1)$ and $T_2 = (Q_2, L_2, O, H_2)$ be transition systems with the same output space $O$. A relation $R \subseteq Q_1 \times Q_2$ is said to be a simulation relation from $T_1$ to $T_2$ if $(p_1, p_2) \in R$ implies:

1. $H(p_1) = H(p_2)$;
2. $p_1 \xrightarrow{l_1} q_1$ imply the existence of $q_2 \in Q_2$ such that $p_2 \xrightarrow{l_2} q_2$ with $(q_1, q_2) \in R$. 

[Note: There is a typographical error in the definition where the label $l_1$ should be $l_2$.]
Key ingredients
Approximate (bi)simulation

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Relation $R$ is said to be a bisimulation relation between $T_1$ and $T_2$ if, in addition to 1. and 2., $(p_1, p_2) \in R$ also implies:

3. $p_2 \xrightarrow{l_2} q_2$ imply the existence of $q_1 \in Q_1$ such that $p_1 \xrightarrow{l_1} q_1$ with $(q_1, q_2) \in R$. 
Key ingredients
Approximate (bi)simulation

Relaxing the equality constraint $H(p_1) = H(p_2)$ leads to approximate (bi)simulation.

Definition (Girard and Pappas 2005, Tabuada 2005)

Let $T_1 = (Q_1, L_1, \xrightarrow{1}, O, H_1)$ and $T_2 = (Q_2, L_2, \xrightarrow{2}, O, H_2)$ be metric transition systems with the same output space $O$ and let $\varepsilon \in \mathbb{R}^+$. A relation $R \subseteq Q_1 \times Q_2$ is said to be a $\varepsilon$-approximate simulation relation from $T_1$ to $T_2$ if $(p_1, p_2) \in R$ implies:

1. $d(H(p_1), H(p_2)) \leq \varepsilon$;
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Relaxing the equality constraint $H(p_1) = H(p_2)$ leads to approximate (bi)simulation.

**Definition (Girard and Pappas 2005, Tabuada 2005)**

Let $T_1 = (Q_1, L_1, 1, O, H_1)$ and $T_2 = (Q_2, L_2, 2, O, H_2)$ be metric transition systems with the same output space $O$ and let $\varepsilon \in \mathbb{R}^+$. A relation $R \subseteq Q_1 \times Q_2$ is said to be an $\varepsilon$-approximate simulation relation from $T_1$ to $T_2$ if $(p_1, p_2) \in R$ implies:

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A simple idea

\[ \dot{x}_1 = x_2 \]
\[ \dot{x}_2 = u \]

\[ U = \{u_-, u_0, u_+\} \]

\[ u_-(t) = -1 \quad \forall t \in [0, 1] \]
\[ u_0(t) = 0 \quad \forall t \in [0, 1] \]
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Can we extrapolate from this finite transition system?
Key ingredients
A simple idea

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\end{align*}
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Yes, provided that we know how to robustify it!
Key ingredients
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Yes, provided that we know how to robustify it!
Key ingredients
Incremental stability

**Definition (δ-GAS)**

A control system $\Sigma$ is *incrementally globally asymptotically stable* (δ–GAS) if it is forward complete and there exist a $\mathcal{KL}^a$ function $\beta$ such that for any $t \in \mathbb{R}_0^+$, any $x, y \in \mathbb{R}^n$ and any $u \in U$ the following condition is satisfied:

$$
\|x(t, x, u) - x(t, y, u)\| \leq \beta(\|x - y\|, t).
$$

---

A continuous function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to belong to class $\mathcal{KL}_\infty$ if, for each fixed $s$, the map $\beta(r, s)$ is strictly increasing, $\beta(0, s) = 0$ and $\beta(r, s) \to \infty$ as $r \to \infty$, and for each fixed $r$, the map $\beta(r, s)$ is decreasing with respect to $s$ and $\beta(r, s) \to 0$ as $s \to \infty$. 
Key ingredients

Incremental stability

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A control system Σ is *incrementally globally asymptotically stable* (δ–GAS) if it is forward complete and there exist a $\mathcal{KL}^a$ function $\beta$ such that for any $t \in \mathbb{R}_0^+$, any $x, y \in \mathbb{R}^n$ and any $u \in U$ the following condition is satisfied:

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Definition (δ-GAS)

A control system Σ is incrementally globally asymptotically stable (δ–GAS) if it is forward complete and there exist a $\mathcal{KL}^a$ function $\beta$ such that for any $t \in \mathbb{R}_0^+$, any $x, y \in \mathbb{R}^n$ and any $u \in \mathcal{U}$ the following condition is satisfied:

$$\|x(t, x, u) - x(t, y, u)\| \leq \beta(\|x - y\|, t).$$

A continuous function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ is said to belong to class $\mathcal{KL}_{\infty}$ if, for each fixed $s$, the map $\beta(r, s)$ is strictly increasing, $\beta(0, s) = 0$ and $\beta(r, s) \rightarrow \infty$ as $r \rightarrow \infty$, and for each fixed $r$, the map $\beta(r, s)$ is decreasing with respect to $s$ and $\beta(r, s) \rightarrow 0$ as $s \rightarrow \infty$. 
A control system $\Sigma$ is \textit{incrementally globally asymptotically stable} ($\delta$–GAS) if it is forward complete and there exist a $\mathcal{KL}^a$ function $\beta$ such that for any $t \in \mathbb{R}_{0}^+$, any $x, y \in \mathbb{R}^n$ and any $u \in \mathcal{U}$ the following condition is satisfied:

$$
\|x(t, x, u) - x(t, y, u)\| \leq \beta(\|x - y\|, t).
$$

\(^a\)A continuous function $\beta : \mathbb{R}_{0}^+ \times \mathbb{R}^+_0 \rightarrow \mathbb{R}^+$ is said to belong to class $\mathcal{KL}_\infty$ if, for each fixed $s$, the map $\beta(r, s)$ is strictly increasing, $\beta(0, s) = 0$ and $\beta(r, s) \rightarrow \infty$ as $r \rightarrow \infty$, and for each fixed $r$, the map $\beta(r, s)$ is decreasing with respect to $s$ and $\beta(r, s) \rightarrow 0$ as $s \rightarrow \infty$. 
A control system $\Sigma$ is *incrementally input–to–state stable* ($\delta$–ISS) if it is forward complete and there exist a $\mathcal{KL}$ function $\beta$ and a $\mathcal{K}\infty$ function $\gamma$ such that for any $t \in \mathbb{R}_0^+$, any $x, y \in \mathbb{R}^n$ and any $u, v \in \mathcal{U}$ the following condition is satisfied:

$$
\|x(t, x, u) - x(t, y, v)\| \leq \beta(\|x - y\|, t) + \gamma(\|u - v\|_{\infty}).
$$

*\footnote{A continuous function $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to belong to class $\mathcal{K}\infty$ if $\gamma$ is strictly increasing, $\gamma(0) = 0$ and $\gamma(r) \to \infty$ as $r \to \infty$.}
Key ingredients
Incremental stability

Definition (δ-ISS)
A control system Σ is incrementally input–to–state stable (δ–ISS) if it is forward complete and there exist a KL function β and a $\mathcal{K}_\infty^a$ function γ such that for any $t \in \mathbb{R}_0^+$, any $x, y \in \mathbb{R}^n$ and any $u, v \in \mathcal{U}$ the following condition is satisfied:

$$\|x(t, x, u) - x(t, y, v)\| \leq \beta(\|x - y\|, t) + \gamma(\|u - v\|_{\infty}).$$

$a$A continuous function $\gamma : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ is said to belong to class $\mathcal{K}_\infty$ if $\gamma$ is strictly increasing, $\gamma(0) = 0$ and $\gamma(r) \rightarrow \infty$ as $r \rightarrow \infty$. 
Key ingredients
Incremental stability

1. For linear control systems, that is, $\dot{x} = Ax + Bu$, both $\delta$-GAS and $\delta$-ISS are equivalent to stability of $A$ (all the eigenvalues of $A$ have negative real part);

2. In the nonlinear case, by restricting attention to a compact set, GAS implies $\delta$-GAS and ISS implies $\delta$-ISS;

3. Both $\delta$-GAS and $\delta$-ISS admit Lyapunov characterizations.
Main results
Quantization of control systems

Given a control system $\Sigma = (\mathbb{R}^n, U, \mathcal{U}, f)$, a time quantization $\tau \in \mathbb{R}^+$, a space quantization $\eta \in \mathbb{R}^+$, and an input quantization $U_\tau \subseteq \mathcal{U}$, define the transition system:

$$T_{\eta U_\tau} (\Sigma) := (Q_{\eta U_\tau}, L, \eta U_\tau, O, H),$$

where:

- $Q_{\eta U_\tau} = [\mathbb{R}^n]_\eta$;
- $L = U_\tau$;
- $q \xrightarrow{\eta U_\tau}^u p$ if $\|x(\tau, q, u) - p\| \leq \frac{\eta}{2}$;
- $O = \mathbb{R}^n$;
- $H = 1_{\mathbb{R}^n}$. 
Main results

Existence of approximate simulations

**Theorem**

Let $\Sigma$ be a control system and let $\varepsilon \in \mathbb{R}^+$ be any desired precision. If $\Sigma$ is $\delta$-GAS, then for any $\tau \in \mathbb{R}^+$, for any $U_\tau \subseteq U$, and for any $\eta \in \mathbb{R}^+$ satisfying the following inequality:

$$\beta(\varepsilon, \tau) + \frac{\eta}{2} \leq \varepsilon$$

there exists an $\varepsilon$-approximate simulation relation $R$ from $T_{\eta U_\tau}(\Sigma)$ to $T_\tau(\Sigma)$ satisfying $R(Q_{\eta U_\tau}) = \mathbb{R}^n$. 

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Main results
Existence of approximate simulations

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Let $\Sigma$ be a control system and let $\varepsilon \in \mathbb{R}^+$ be any desired precision. If $\Sigma$ is $\delta$-GAS, then for any $\tau \in \mathbb{R}^+$, for any $\mathcal{U}_\tau \subseteq \mathcal{U}$, and for any $\eta \in \mathbb{R}^+$ satisfying the following inequality:

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there exists an $\varepsilon$-approximate simulation relation $R$ from $T_{\eta \mathcal{U}_\tau}(\Sigma)$ to $T_{\tau}(\Sigma)$ satisfying $R(Q_{\eta \mathcal{U}_\tau}) = \mathbb{R}^n$. 
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![Diagram of a control system with approximate simulation relations](image)
Main results
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Let $\Sigma$ be a control system and let $\varepsilon \in \mathbb{R}^+$ be any desired precision. If $\Sigma$ is $\delta$-GAS, then for any $\tau \in \mathbb{R}^+$ for any $U_\tau \subseteq U$, and for any $\eta \in \mathbb{R}^+$ satisfying the following inequality:

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there exists an $\varepsilon$-approximate simulation relation $R$ from $T_{\eta U_\tau}(\Sigma)$ to $T_\tau(\Sigma)$ satisfying $R(Q_{\eta U_\tau}) = \mathbb{R}^n$.

Under the assumptions of the previous theorem, there exists a countable set of control quanta $U_\tau$ rendering $T_{\eta U_\tau}(\Sigma)$ $\varepsilon$-approximate bisimilar to $T_\tau(\Sigma)$. 
Main results
Existence of approximate simulations

Theorem

Let $\Sigma$ be a control system and let $\varepsilon \in \mathbb{R}^+$ be any desired precision. If $\Sigma$ is $\delta$-GAS, then for any $\tau \in \mathbb{R}^+$ for any $U_\tau \subseteq U$, and for any $\eta \in \mathbb{R}^+$ satisfying the following inequality:

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there exists an $\varepsilon$-approximate simulation relation $R$ from $T_{\eta U_\tau}(\Sigma)$ to $T_\tau(\Sigma)$ satisfying $R(Q_{\eta U_\tau}) = \mathbb{R}^n$.

Under the assumptions of the previous theorem, there exists a countable set of control quanta $U_\tau$ rendering $T_{\eta U_\tau}(\Sigma)$ $\varepsilon$-approximate bisimilar to $T_\tau(\Sigma)$.

But how do we compute $U_\tau$?
Main results
Existence of approximate bisimulations

Theorem

Let $\Sigma$ be a digital control system and let $\varepsilon \in \mathbb{R}^+$ be any desired precision. If $\Sigma$ is $\delta$-ISS, then for any $\tau \in \mathbb{R}^+$, for $U(\tau) = [U]_\mu$, and for any $\eta \in \mathbb{R}^+$ satisfying the following inequality:

$$
\beta(\varepsilon, \tau) + \frac{\eta}{2} + \gamma(\mu) \leq \varepsilon
$$

there exists an $\varepsilon$-approximate bisimulation relation $R$ between $T_{\eta U \tau}(\Sigma)$ and $T_{\tau}(\Sigma)$. 
Example
A non-holonomic robot

Let us consider the simplest nonholonomic robot:

\[
\begin{align*}
\dot{x} &= v \cos \theta \\
\dot{y} &= v \sin \theta \\
\dot{\theta} &= \omega
\end{align*}
\]

(1) (2) (3)

and construct a finite abstraction by working on \([-2, 2] \times [-2, 2] \times [0, 2\pi]\) and by considering constant input curves of duration 3s and assuming values on \(\{0, 1\} \times \{-1.1, -1, 1, 1.1\}\).
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Example
A non-holonomic robot

Periodic orbits: find a periodic orbit passing through the origin.
Example

A non-holonomic robot

Periodic orbits: find a periodic orbit passing through the origin.

Searching on the discrete abstraction we obtain:

\[(0, 0, 0.55\pi) \xrightarrow{(1,-1.1)} (1.73, 0, 1.47\pi) \xrightarrow{(1,-1)} (0, 0, 0.55\pi)\]
Example
A non-holonomic robot

Language specifications: execute periodic orbits according to the sequence

right→right→left→left→left→right→right→left→left→left.
Example
A non-holonomic robot

Language specifications: execute periodic orbits according to the sequence

right→right→left→left→left→right→right→left→left→left.
Example
A non-holonomic robot

Switching specifications.

rotate clockwise

rotate counter-clockwise

either clockwise or counter-clockwise
Example
A non-holonomic robot

Switching specifications.

rotate clockwise

rotate counter-clockwise

either clockwise or counter-clockwise
Example
A non-holonomic robot

Switching specifications.

rotate clockwise

rotate counter-clockwise

either clockwise or counter-clockwise
Example
A non-holonomic robot

Interaction with discrete signals.
Putting the pieces together
Abstraction

- The abstraction step can be done for a reasonable class of control systems;
- Recent results eliminate the stability assumption by ensuring only the existence of approximate alternating simulation relations;
- Multi-resolution quantization and other techniques can be used to reduce the size of the abstractions.
Putting the pieces together
Synthesis of discrete controllers

- Synthesis of controllers based on language specifications can be done by resorting to supervisory control or algorithmic game theory;
- Synthesis of controllers based on (bi)simulation specifications can be done by modifying existing algorithms for the construction of (bi)simulations;
- Incorporating timing considerations is crucial to address real-time issues.
The refinement of discrete to hybrid controllers is based on the approximate bisimulation relation between the discrete abstraction and the continuous plant;

The hybrid controller is a formal model for embedded code. Automated code generation from this model is conceptually simple;

Correctness of operation depends on real-time scheduling and many other considerations.
Many questions remain open and much work is to be done before we can synthesize correct-by-design embedded control software.

The ingredients, however, are becoming available.
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The ingredients, however, are becoming available.

The work described in this talk was the result of:

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