Performance Bounds for Constrained Linear Stochastic Control

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Outline

• constrained linear stochastic control problem

• the linear quadratic case

• performance bound

• a suboptimal control scheme based on performance bound

• numerical examples
Linear stochastic system

• linear dynamical system with process noise:

\[ x_{t+1} = Ax_t + Bu_t + w_t, \quad t = 0, 1, \ldots, \]

– \( x_t \in \mathbb{R}^n \) is the state
– \( u_t \in \mathcal{U} \) is the control input
– \( \mathcal{U} \subset \mathbb{R}^m \) is the input constraint set, with \( 0 \in \mathcal{U} \)
– \( w_t \in \mathbb{R}^n \) is zero mean IID process noise, \( \mathbb{E} w_t w_t^T = W \)

• state feedback control policy:

\[ u_t = \phi(x_t), \quad t = 0, 1, \ldots, \]

\( \phi : \mathbb{R}^n \rightarrow \mathcal{U} \) is the state feedback function
Objective

• objective is average stage cost:

\[
J = \limsup_{T \to \infty} \frac{1}{T} \mathbb{E} \sum_{t=0}^{T-1} (\ell_x(x_t) + \ell_u(u_t))
\]

- \(\ell_x : \mathbb{R}^n \to \mathbb{R}\) is state stage cost function
- \(\ell_u : \mathcal{U} \to \mathbb{R}\) is the input state cost function

• \(\ell_x, \ell_u, \mathcal{U}\) need not be convex
**Stochastic control problem**

- stochastic control problem: *choose feedback function* $\phi$ *to minimize* $J$

- infinite dimensional nonconvex optimization problem

- problem data:
  - dynamics and input matrices $A$, $B$
  - distribution of process noise $w_t$
  - state and input cost functions $\ell_x$, $\ell_u$
  - input constraint set $\mathcal{U}$

- $\phi^*$ denotes an optimal feedback function

- $J^*$ denotes optimal objective value
‘Solution’ via dynamic programming

• find $V^* : \mathbb{R}^n \rightarrow \mathbb{R}$ and $\alpha$ with

$$V^*(z) + \alpha = \min_{v \in \mathcal{U}} (\ell_u(v) + \mathbf{E} V^*(Az + Bu + w_t))$$

(expectation is over $w_t$)

• optimal feedback function is then

$$\phi^*(z) = \arg\min_{v \in \mathcal{U}} (\ell_u(v) + \mathbf{E} V^*(Az + Bu + w_t))$$

• optimal value of stochastic control problem is $J^* = \alpha$
Stochastic control problem

• generally very hard to solve
  (even more: how would we represent a general function \( \phi \)?)

• can be effectively solved
  – when the problem dimensions are very small, \( e.g., n = m = 1 \)
  – when \( \mathcal{U} = \mathbb{R}^m \) and \( \ell_x, \ell_u \) are convex quadratic

• many suboptimal methods have been proposed
  – can evaluate \( J \) for a given \( \phi \) via Monte Carlo simulation
  – but how suboptimal is it?

• this talk: *an effective method for finding a (good) lower bound on \( J^* \)*
Control-Lyapunov policy

- control-Lyapunov policy is

\[ \phi_{\text{clf}}(z) = \arg\min_{v \in \mathcal{U}} (\ell_u(v) + \mathbf{E} V_{\text{clf}}(Az + Bv + w_t)) \]

- \( V_{\text{clf}} : \mathbb{R}^n \rightarrow \mathbb{R} \) (which is to be chosen) is the control-Lyapunov function
- when \( V_{\text{clf}} = V^* \), this is optimal policy

- when \( V_{\text{clf}} \) is quadratic, the control-Lyapunov policy simplifies to

\[ \phi_{\text{clf}}(z) = \arg\min_{v \in \mathcal{U}} (\ell_u(v) + V_{\text{clf}}(Az + Bv)) \]

since \( \mathbf{E} w_t = 0 \), and term involving \( \mathbf{E} w_t w_t^T = W \) is constant
The performance bound

our method:

• computes a lower bound $J^{lb} \leq J^*$ using convex optimization (hence is tractable)

• bound is computed for each specific problem instance

• (at this time) cannot guarantee tightness of bound
**Unconstrained linear quadratic control**

- can effectively solve stochastic control problem when
  - $\mathcal{U} = \mathbb{R}^m$ (no constraints)
  - $\ell_x(z) = z^T Q z$, $\ell_u(v) = v^T R v$, $Q \succeq 0$, $R \succeq 0$

- optimal cost is $J^*_{lq} = \text{Tr}(P^*_lq W)$

- optimal state feedback function is $\phi^*(z) = K^*_lq z$, where
  \[
  K^*_lq = -(R + B^T P^*_lq B)^{-1} B^T P^*_lq A
  \]

- $P^*_lq$ is positive semidefinite solution of ARE
  \[
  P^*_lq = Q + A^T P^*_lq A - A^T P^*_lq B (R + B^T P^*_lq B)^{-1} B^T P^*_lq A
  \]
Linear quadratic control via LMI/SDP

• can characterize $J_{lq}^*$ and $P_{lq}^*$ via the semidefinite program (SDP)

$$\begin{align*}
\text{maximize} & \quad \text{Tr}(PW) \\
\text{subject to} & \quad P \succeq 0 \\
& \quad \begin{bmatrix}
R + B^T P B & B^T P A \\
A^T P B & Q + A^T P A - P
\end{bmatrix} \succeq 0
\end{align*}$$

- variable is $P$
- optimal point is $P = P_{lq}^*$; optimal value is $J_{lq}^*$

• solution does not depend on $W$, as long as $W \succ 0$

• constraints are convex in $(P, Q, R)$, so $J_{lq}^*(Q, R)$ is a concave function of $(Q, R)$
Basic bound

- suppose $Q \succeq 0$, $R \succeq 0$, $s$ satisfy

$$z^T Qz + v^T Rv + s \leq \ell_x(z) + \ell_u(v) \quad \text{for all } z \in \mathbb{R}^n, v \in \mathcal{U}$$

i.e., quadratic stage costs are everywhere smaller than $\ell_x + \ell_v$

- then $J^*_1(Q, R) + s$ is a lower bound on $J^*$

- follows from monotonicity of stochastic control cost w.r.t. stage costs

- lefthand side is optimal value of unconstrained quadratic problem
Optimizing the bound

- can optimize the lower bound over $Q$, $R$, $s$ by solving

$$\begin{align*}
\text{maximize} & \quad J^*_l(Q, R) + s \\
\text{subject to} & \quad Q \succeq 0, \quad R \succeq 0, \\
& \quad z^T Q z + v^T R v + s \leq \ell_x(z) + \ell_u(v) \quad \text{for all } z \in \mathbb{R}^n, \quad v \in \mathcal{U}
\end{align*}$$

- a convex optimization problem
  - objective is concave
  - constraints are convex
  - last constraint is convex in $Q$, $R$, $s$ for each $z$ and $v$

- last constraint is semi-infinite, parameterized by the (infinite) set $z \in \mathbb{R}^n$, $u \in \mathcal{U}$
Numerical examples

• illustrate bounds for 3 examples
  – small problem with trilevel inputs
  – large problem with box constraints
  – discretized mechanical control system

• compare lower bound with various heuristic policies
  – projected linear state feedback
  – model predictive control
  – control-Lyapunov policy
Small problem with trilevel inputs

- $n = 8$, $m = 2$

- $A$, $B$ matrices randomly generated; $A$ scaled so $|\lambda_i(A)| = 1$

- Quadratic stage costs with $R_0 = I$, $Q_0 = I$

- $w_t \sim \mathcal{N}(0, 0.25I)$

- Finite input set: $\mathcal{U} = \{-0.2, 0, 0.2\}^2$
Large problem with box constraints

- $n = 30, \ m = 10$

- $A, B$ matrices randomly generated; $A$ scaled so $|\lambda_i(A)| = 1$

- quadratic stage costs with $R_0 = I, \ Q_0 = I$

- $w_t \sim \mathcal{N}(0, 0.25I)$

- box input constraints: $\mathcal{U} = \{v \in \mathbb{R}^m \mid \|v\|_\infty \leq 0.1\}$
Discretized mechanical control system

- 6 masses connected by springs; 3 input tensions between masses
- Quadratic stage costs with $R_0 = I$, $Q_0 = I$
- $w_t$ uniform on $[-0.5, 0.5]$
- Box input constraints: $\mathcal{U} = \{v \in \mathbb{R}^m \mid \|v\|_\infty \leq 0.1\}$
Heuristic policies

- projected linear state feedback with $K_{pl} = K_{lq}^*$

- control-Lyapunov policy with $V_{clf}(z) = z^T P_{lb} z$

- model predictive control (MPC) with $T = 30$, $V_{mpc}(z) = z^T P_{lb} z$

(for trilevel example we solve convex relaxation with $u(t) \in [-0.2, 0.2]$, then round value to $\{-0.2, 0, 0.2\}$)
### Results

<table>
<thead>
<tr>
<th></th>
<th>small trilevel</th>
<th>large random</th>
<th>masses</th>
</tr>
</thead>
<tbody>
<tr>
<td>PLSF</td>
<td>12.9</td>
<td>31.3</td>
<td>269.8</td>
</tr>
<tr>
<td>CLF</td>
<td>10.8</td>
<td>25.6</td>
<td>61.1</td>
</tr>
<tr>
<td>MPC</td>
<td>10.9</td>
<td>25.7</td>
<td>58.9</td>
</tr>
<tr>
<td>$J_{lb}$</td>
<td>9.1</td>
<td>23.8</td>
<td>43.2</td>
</tr>
</tbody>
</table>

- control-Lyapunov with $P_{lb}$ and MPC achieve similar performance.
- control-Lyapunov policy can be computed *very* fast (in tens of microseconds); MPC policy can be computed in milliseconds.
- bound $J_{lb}$ is reasonably close to $J$ for these examples.
Conclusions

• we’ve shown how to find lower bounds on optimal performance for constrained linear stochastic control problems

• requires solution of convex optimization problem, hence is tractable

• provides only provable lower bound on optimal performance that we are aware of

• as a by-product, provides excellent choice for quadratic control-Lyapunov function

• in many cases, gives everything you want:
  – a provable lower bound on performance
  – a relatively simple heuristic policy that comes close
References


- Wang & Boyd, *Performance Bounds and Suboptimal Policies for Linear Stochastic Control via LMIs*, (manuscript)

  (similar results in MDP setting)